A NEW AND COMPLETE

TREATISE

OF

SPHERICAL TRIGONOMETRY:

In which are contained the

ORTHOGRAPHIC, ANALYTICAL, and LOGARITHMICAL

SOLUTIONS

OF THE SEVERAL CASES OF

SPHERICAL TRIANGLES,

Whether right-angled or oblique;

A COMPREHENSIVE THEORY of the FLUXIONS of these TRIANGLES;

AND

A Variety of CURIOUS and INTERESTING Particulars not to be met with in any other TREATISE upon this Subject.

CAREFULLY TRANSLATED FROM THE FRENCH OF

MR. MAUDUIT,

By W. CRAKELT.

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MDCCXCI.

TO

JOHN BEVIS, M. D.

FELLOW OF THE ROYAL SOCIETY,

AND

OF THE ROYAL ACADEMY OF SCIENCES AT BERLIN,

THIS TRANSLATION

IS INSCRIBED BY

HIS MUCH OBLIGED

AND OBEDIENT SERVANT,

W. CRAKELT.

PREFACE.

MOST of the Treatiſes of ſpherical Trigonometry, that have been hitherto publiſhed, contain only directions for ſolving ſpherical Triangles *(Logarithmically)* by proportions; and none, that I know of, hath united the different ſolutions whereof Problems in this part of the Mathematics are ſuſceptible.——Now, with a very little attention to the nature of the ſubject, it will appear that the ſolutions may be of three kinds, viz. 1°, ſuch as reſpect *Analogies* (of which kind are thoſe given in all the common Treatiſes of ſpherical Trigonometry); 2°, ſuch as are obtained by the ſcale and compaſs from *Projections* or *Geometrical Conſtructions*, and 3°, ſuch as are deduced from projections by the application of *Algebra*; all which will be treated of at large in the following Work.

In the firſt Chapter I have laid down the definitions and fundamental principles of *ſines*, *coſines*, *tangents*, *cotangents*, &c. in a great variety of particular formulæ, as well as infinite ſeries. I have likewiſe inſerted a ſhort Theory upon the *true* nature of *Imaginaries* (ſince they have been frequently uſed

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in ſeries relative to the different powers of *ſines* and *coſines* of multiple arcs), and, from an attempt to diſcover the Factors of the binomial $aa + bb$, have eaſily inferred, that theſe expreſſions are only ſigns of an abſurd ſuppoſition into which we neceſſarily fall, whenever we regard that as the product of two quantities which in reality is not. But, for what I have ſaid upon this head, I acknowledge myſelf indebted to the late ingenious, and much regretted, Mr. *Boiſelou.* ——Some may perhaps conſider this part of the Work as a diſplay of algebraical formulæ, entirely uſeleſs in a Treatiſe of Trigonometry: however, from ſuch I ſhall take the liberty of requeſting a ſuſpenſion of judgment, at leaſt till they have undertaken the ſtudy of the different works of calculation relative to the theory of *Newton:* for Aſtronomy is become ſo excellent and delicate a ſcience, that it is impoſſible to have too many helps for facilitating the calculations it requires, and perfecting the ſeveral parts upon which it depends; and for thoſe Gentlemen, who have no intention of ſtudying this ſubject thoroughly, a peruſal of the five or ſix firſt Theorems may be ſufficient.

In the ſecond Chapter I have demonſtrated the principal properties of ſpherical triangles; premiſing, in the firſt of the four Sections into which this Chapter is divided, the neceſſary definitions, *&c.*——The ſecond Section

tion treats of the resolution of right-angled spherical triangles; and, as this part is of almost perpetual use, I have attempted to give it the greatest simplicity whereof it can be susceptible; for every Theorem requires only a single analogy for its demonstration: and, to enable the memory still more to retain the different solutions, I have added a demonstration of the general Theorem of *Neper*, which reduces the sixteen cases of right-angled triangles to two, and requires only the bare enumeration of these cases to be verified.——The third Section contains the resolution of all the cases of oblique-angled spherical triangles. Herein I have inserted a Theorem analogous to the Theorem of *Neper* for right-angled triangles, and the same with that which Mr. *Pingré* hath given in the *Memoirs of the Academy*, except for a few trifling alterations, which I thought necessary to be made in order to render it somewhat more simple. I have likewise superadded to the solutions already given, for the case of three sides or three angles, several others which I take to be new; and which, when applied to right-lined triangles, furnish us with solutions much more easy and compendious than any to be met with in common Treatises upon Trigonometry.——In the last Section complete demonstrations of the famous Analogies or Theorems of *Neper* (which have been considerably mutilated and altered in

in the Courſe of Mr. *Wolf)* are given; and ſuch as carry along with them ſeveral other Theorems equally general, whereof Mr. *Simpſon* hath ſpecified only particular caſes. To this Section I have annexed three Tables containing the ſubſtance of the whole Work, at leaſt as far as the more ordinary practice extends, and have expreſſed them in words inſtead of letters, that miſapprehenſions and miſtakes in calculations might be avoided as much as poſſible. The firſt Table is for the reſolutions of right-angled ſpherical triangles, and the ſecond and third for thoſe of oblique-angled ones; but the laſt is, by means of the analogies of *Neper*, equally applicable to right-lined triangles, and may moreover have this advantage attending it; that, on account of the perfect reſemblance prevailing between the ſolutions of ſimilar caſes of theſe two kinds of triangles, a perſon, previouſly acquainted with the proportions for plain triangles, may learn the principal parts of ſpherical Trigonometry in a quarter of an hour. ——As this Chapter contains all that is eſſential and neceſſary to the reſolution of ſpherical triangles, they, who would make themſelves maſters of this part of Trigonometry only, may paſs from hence immediately to the ſixth Chapter, in order to ſee the method of putting its formulæ into Logarithms.

The third Chapter contains the Geometrical or Graphical ſolutions of the different

caſes

cafes that had been before folved by analogies. I have, in prefcrence to the reft, made ufe of the *orthographic* projections, whereof Aftronomers fuppofe at every inftant a complete theory: however, I have faid fomething upon the *ftereographic* projections alfo, on account of the great and frequent ufe which is made of them in Maps; and have, laftly, fubjoined a new folution eafily applicable to the feveral cafes before fpecified, and deduced from the expanfion of the parts of the triangle under confideration.

The fourth Chapter contains an application of the Algebraical analyfis to the Geometrical conftructions in the preceding Chapter. In this part it is that, by the help of calculation, the folutions for all particular cafes are (if I may be allowed the expreffion) exhaufted. Moft of the analogies of the fecond Chapter are deduced from general cafes in this by fimple inferences: a variety of new formulæ is added to thofe that had been before given; and the apparent difference between the fynthetic and analytic folutions for the fame cafes perfectly reconciled. And, that nothing might be wanting, I have fhewn the method of conftructing the moft complicated of the formulæ by the Logarithms: after which I have given the folutions of equations of the fecond and third degree by the Tables of *fines*, &c. a thing that may prove to be of the greateft utility, when com-

mensurable roots cannot be obtained. The resolutions of equations of the third degree were communicated to me by a gentleman already well known in the world by different works, and yet less estimable on account of his profound learning than the excellence of his character: to him I likewise owe the new solution of the Problem concerning the shortest twilight, inserted in the fifth Chapter,

Which is little more than a Translation of that excellent Treatise of Mr. *Cotes*: "*De* "*æstimatione errorum in mixtâ Mathesi*," containing an application of the modern analysis or the calculation of Fluxions * to both plain and spherical Trigonometry. What relates to right-lined triangles I have deduced from the formulæ for spherical triangles, that I might render the Theory as simple as possible, and abridge in some measure a Work that was already become much longer than I at first intended: and, in the last place, have sufficiently elucidated this Theory by the application of it to different examples.

In the sixth Chapter I have shewn the method of constructing by the Logarithms the formulæ given in the second Chapter; and

* In this Chapter the Translator has taken the liberty of changing the *Differential* method of consideration into the *Fluxional*, and also of supplying such references and comparisons as appeared to him necessary and conducive to the more speedy comprehension of several of its parts.

this

this I have done in a ſelect collection of Aſtronomical Problems, through a perſuaſion that ſolutions derived from triangles having ſome particular denominations, could not but prove much more intereſting to Learners, than ſuch as were derived from triangles abſtractedly conſidered.

The motive which firſt induced me to undertake the writing of this Treatiſe aroſe from hence: Having had occaſion to teach the Elements of Aſtronomy of the late Abbé *De la Caille*, I found, upon a peruſal of the little Treatiſe of Trigonometry prefixed to that Work, a great number of formulæ without demonſtrations, and therefore, well knowing that Learners are generally deſirous of making themſelves perfect as they proceed in the ſtudy of Mathematical truths, I thought that I ſhould do them no inconſiderable piece of ſervice in giving them all requiſite aſſiſtance in ſuch difficulties as few would otherwiſe be able to ſurpaſs. How well I have ſucceeded in my deſign, as alſo in the execution of the other parts of the Work, I leave to the Public to determine: however, ſhould it be found deſerving of their encouragement, it ſhall be followed by another Treatiſe upon Dialling as its ſequel. 'Tis particularly in this part of the Mathematics that ſpherical Trigonometry is advantageouſly applied: for the deſcriptions of Dials by lines are not ſufficiently exact to deſerve any

certain confidence; and there hath been yet no Treatiſe upon Dialling publiſhed (that I know of) wherein the whole hath been performed by the calculation of ſpherical triangles, though ſuch method could not fail to prove of the utmoſt importance in the practice, as all the different hour-angles, which are uſually calculated by a ſeries of plain triangles, might be then determined without the leaſt relation to one another.

CON-

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E R R A T A.

Page 89, line 8, for *half the ſum* read *the ſum*; p. 104, l. 11, for dD read dG; p. 135, l. 16, for 112 read 212; p. 137, laſt line, for $\frac{f}{mr}$ read $\frac{fgh}{mr}$; p. 158, l. 27, for *ſine* read *ſign*; p. 186, l. 28, for *ſide* read *angle*.

THE PRINCIPLES OF SPHERICAL TRIGONOMETRY.

CHAP. I.

Containing the Definitions and chief Properties of the several lines which are usually considered in Trigonometry; but particularly of such as we shall have more frequent occasion to mention in the course of this Work.

DEFINITIONS.

ART. I. A Right line MP, drawn from one end of an arc AM, or angle ACM, perpendicularly to the radius which passeth through the other end thereof, is called the *Sine* of such arc or angle. *Fig.* 1.

2. From this definition it is evident, that the right line MQ is the *sine* of the arc BM, the

complement of the arc AM; but if this line be considered with relation to the arc AM, it is called ts *Cosine*.

3. A right line AT perpendicular to the end of a radius, and terminated by the radius produced which passeth through the other end M of an arc AM, is called the *Tangent* of this arc.

4. A right line Bt, which is the *tangent* of the arc BM, the complement of AM, when considered with relation to the said arc AM, is called its *Cotangent*.

5. The line intercepted between the centre of the circle and the *tangent* of an arch, is called the *Secant* of that arch: thus CT is the *secant* of the arch AM, and Ct that of the arch BM: but if the *secant* Ct be considered with relation to the arch AM, the complement of the arch BM whereunto it belongs, it will be called the *Cosecant* thereof.

6. That part of the radius contained between the circumference and *sine* of an arc, is called the *Versed-sine* of such arc: thus AP is the *versed-sine* of the arc AM; BQ that of the arc BM; and aP and bQ the *versed-sines* of the obtuse angles aCM, bCM. If the *versed-sine* BQ be considered as belonging to an arc which is the complement of AM, with respect to the said arch AM, it will be called the *Coversed-sine*; and contrarily, AP will be the *coversed-sine* of the arch BM.

COROLLARY.

7. From these definitions it follows, that two angles, which are the supplements to each other, have the same *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, and *versed-sines*. It must be observed however, that the obtuse angle has for its complement the angle of its excess above 90°; which angle being evidently the same with that which the acute angle wants of 90°, must of consequence be regarded as negative.

ADVER-

ADVERTISEMENT.

8. In all the following parts of this work we ſhall denote the radius of the circle by R or r, unleſs we ſuppoſe it equal to unity; which ought ſeldom to be done, that the homogeneity of the terms in the calculations may be preſerved as much as poſſible. We ſhall alſo denote the *ſine* of an arch by *ſin.* its *coſine* by *coſ.* its *tangent* by *tang.* its *cotangent* by *cot.* its *ſecant* by *ſec.* its *coſecant* by *coſec.* its *verſed-ſine* by *verſ-ſin.* and its *coverſed-ſine* by *cover-ſin*: but in the algebraical expreſſions we ſhall put the *ſine* $= s$; *coſine* $= c$; *tangent* $= t$; *cotangent* $= \tau$; *ſecant* $= S$; *coſecant* $= ſ$; *verſed-ſine* $= v$, and *coverſed-ſine* $= u$.

Theorem I.

9. *Let any arch of a circle, as* AM, *be aſſumed, (which we ſhall always denote by* A*), whereof* PM *is the ſine, and* MQ *the coſine;* AT *the tangent, and* Bt *the cotangent; and the coſine of this arch will be to the ſine, as radius is to the tangent*; that is, *Coſ.* A : *ſin.* A : : R : *tang.* A. *Fig.* 1.

DEMONSTRATION.

The ſimilar triangles CPM and CAT give CP : PM : : CA : AT; or, *Coſ.* A : *ſin.* A : : R : *tang.* A. Q. E. D.

Corollary I.

10. The triangles CQM, CBt, being alſo ſimilar, will give CQ : QM : : CB : Bt; or, *Sin.* A : *coſ.* A : : R : *cot.* A.

Corollary II.

11. It follows from hence, that *tang.* A $= \frac{ſin.\,A}{coſ.\,A} \times R$, and *cot.* A $= \frac{coſ.\,A}{ſin.\,A} \times R$; or algebraically, $t = \frac{sr}{c}$, and $\tau = \frac{cr}{s}$; and conſequently, *ſin.* A $\times$ R $=$ *coſ.* A $\times$ *tang.* A, and *coſ.* A $= \frac{ſin.\,A \times cot.\,A}{R}$.

COROLLARY III.

12. Hence it again follows, that the *cotangents* of any number of arcs are always in the inverse ratio of the *tangents* of these arcs; since we have evidently, $t : \tau :: \frac{sr}{c} : \frac{cr}{s} :: \frac{s}{c} : \frac{c}{s}$ *: in short, *tang.* A will be $= \frac{RR}{cot\ A}$; or, *tang.* A $\times$ *cot.* A $=$ R R; whence we may perceive, that the radius is a mean proportional between the *tangent* of an arch and that of its complement; a thing which might likewise be easily demonstrated by means of the similar triangles CAT, CBt†.

THEOREM II.

13. *Let the arch* AM *be still retained, and we shall* Fig. 2. *also have this proportion; As radius is to twice the sine of the arch* AM, *so is the cosine of this arch to the sine of twice the arch*; or, which is the same, R : 2 *sin.* A :: *cos.* A : *sin.* 2A; or *alternando*, R : *cos.* A :: 2 *sin.* A : *sin.* 2 A.

DEMONSTRATION.

The right-angled triangles APC, AQN, having the angle at A common, are similar, and give AC : CP :: AN *or* 2 PA : QN; that is to say, R : *cos.* A :: 2 *sin.* A : *sin.* 2 A. Q. E. D.

If in like manner we would find an expression for the *cosine* of the double arch; the similar triangles CPA, CQO, will give CA : CP :: CO : CQ; from whence, (after putting for CO its value drawn

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* *Or thus.* Since *cot.* A $= \frac{RR}{tang.\ A}$, and *cot.* B $= \frac{RR}{tang.\ B}$; therefore we shall have, *Cot.* A : *cot.* B :: $\frac{RR}{tang.\ A} : \frac{\times RR}{tang.\ B}$ :: *tang.* B : *tang.* A.

† For AT : CA :: CB : Bt; or, *tang.* A : R : : R : *cot.* A; wherefore, *tang.* A $\times$ *cot.* A $=$ R R.

drawn from CP—OP or its equal PT *), we shall immediately get, $CQ = \frac{2\,cos.^2 A - RR}{R}$.

COROLLARY.

14. It follows from hence, (*viz.* by taking the product of the extremes and means in the proportion, $R : cos. A :: 2\,sin. A : sin. 2A$,) that, $\frac{1}{2} sin. 2A = \frac{cos. A \times sin. A}{R}$; and if in this equation we put $\frac{sin. A \times cot. A}{R}$ for *cos.* A we shall otherwise have, $\frac{1}{2} sin. 2A = \frac{sin.^2 A \; cot. A}{RR}$; or, $\frac{1}{2} sin. 2A = \frac{sin.^2 A}{tang. A}$, by substituting for the *cotangent* its value in the *tangent*.

THEOREM III.

15. *The radius is a mean proportional; 1°. between the cosine of an arch and its secant; and 2°. between the sine of the same arch and its cosecant*; and therefore we shall have these two analogies; $Cos. A : R :: R : sec. A$, and $Sin. A : R :: R : cosec. A$. Fig. 3.

DEMONSTRATION.

The similar triangles CPM, CAT, and CB*t* give 1°. $CP : CA :: CM : CT$; or, $Cos. A : R :: R : sec. A$; and 2°. $PM : CM :: CB : Ct$; or, $Sin. A : R :: R : cosec. A$. Q. E. D.

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* That OP is = PT, may be very easily proved: for by letting fall the perpendicular PS, it will appear that since AP = PN, therefore QS = SA, and consequently OP = PT. This being granted, the value of CO will be obtained as follows: Since the angle CAT is right, PT or OP will be equal $\frac{\overline{PA}^2}{CP} = \frac{\overline{CA}^2 - \overline{CP}^2}{CP}$, and $CO = CP - \frac{\overline{CA}^2 - \overline{CP}^2}{CP} = cos. A - \frac{RR - cos.^2 A}{cos. A} = \frac{2\,cos.^2 A - RR}{cos. A}$.

COROL-

COROLLARY I.

16. It follows from hence, that $\frac{sec.\ A}{cosec.\ A} \times R = tang.$ A; for we have already seen in *art.* 11. that *tang.* A $= \frac{sin.\ A}{cos.\ A} \times R$; and this quantity is by the present Theorem equal to the quotient $\frac{sec.\ A}{cosec.\ A}$ multiplied by the radius.

COROLLARY II.

17. It moreover follows from hence, that the *sines* of any two arcs A and B will be reciprocally proportional to the *cosecants* of these arcs; and the *cosines* to their *secants*. For since, *cosec.* A $= \frac{RR}{sin.\ A}$, *cosec.* B will also be $= \frac{RR}{sin.\ B}$; likewise, since *sec.* A $= \frac{RR}{cos.\ A}$, we shall have *sec.* B $= \frac{RR}{cos.\ B}$, and consequently these two analogies; *Cosec.* A : *cosec.* B : : $\frac{RR}{sin.\ A} : \frac{RR}{sin.\ B}$: : *sin.* B : *sin.* A; and, *Sec.* A : *sec.* B : : $\frac{RR}{cos.\ A} : \frac{RR}{cos.\ B}$: : *cos.* B : *cos.* A.

THEOREM IV.

18. *Let the arch* AM *and its complement* BM *be divided into two equal parts at the points* K *and* k, *and let the lines* CKl, CkL, *be drawn till they meet the tangents* AT *and* Bt *produced, as far as necessary, in* l *and* L; and we shall by this means have, 1°. *Sec.* A $=$ *cot.* $\frac{1}{2}$ *comp.* A — *tang.* A, and 2°. *Cosec.* A $=$ *cot.* $\frac{1}{2}$ A — *cot.* A.

Fig. 3.

DEMONSTRATION.

The right-angled triangles CB*o* and CAL being similar, on account of the parallels B*o* and CA, will have the angles ALC, BC*o*, equal; but BC*o* is $=$ *o*CM, by construction; wherefore the triangle CTL is isosceles, and consequently LT $=$ CT; but

but $LT = AL - AT$; $AL = cot.\ \frac{1}{2}\ comp.\ AM$, and $AT = tang.\ AM$; and therefore CT, or *Sec.* $A = cot.\ \frac{1}{2}\ comp.\ A - tang.\ A$. Q. E. 1°. D.

2. The right-angled triangles CAO and CBl being ſimilar, on account of the parallels CA and Bl, will have the angles BlC and ACO equal; but ACO is = MCO, by conſtruction; therefore the triangle Ctl is iſoſceles, and the *coſecant* $Ct = tl$: but it is evident that $tl = Bl - Bt = cot.\ \frac{1}{2}A - cot.\ A$; and conſequently the *Coſecant* Ct of the arch AM is equal to the ſame quantity. Q. E. 2°. D.

Theorem V.

19. *From the point* M *to the extremities* b *and* a *of the diameters* Bb, Aa, *let the lines* Mb, Ma, *be drawn, cutting the radii* CA *and* CB *in the points* G *and* g; *moreover let there be drawn through the ſame point* M *the tangent* θMτ, *terminated by the radii* CA *and* CB *produced, as far as neceſſary, at* θ *and* τ, *and by that means determining the lines* Cθ *and* Cτ *reſpectively equal to the ſecant and coſecant of the arch* AM; then we ſhall have, 1°. $Sec.\ A = tang.\ A + tang.\ \frac{1}{2}\ comp.\ A$, and 2°. $Coſec.\ A = cot.\ A + tang.\ \frac{1}{2}A$. *Fig.* 3.

DEMONSTRATION.

The angle bMθ, which is formed by the *tangent* Mθ and chord Mb, is meaſured by half the arch MAb compriſed between its ſides; alſo the angle MGθ, which hath its vertex within the circumference, is meaſured by half the ſum of the arcs ab, AM, contained between its ſides; but ab is = Ab, and therefore the angle MGθ equal to the angle GMθ; conſequently the triangle GθM is iſoſceles, and $G\theta = M\theta = tang.\ A$. Moreover, the angle G$b$C, which is at the circumference, is only half the angle BCM, which hath its vertex at the centre and ſtands upon the ſame arc; therefore, if we regard the radius Cb as the ſine total, CG will

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be the *tangent* of half the complement of the arch A M ; and conſequently, *Sec.* $C\theta = CG + G\theta = tang.\ \frac{1}{2}\ comp.\ A + tang.\ A$†. Q. E. 1°. D.

2°. We might prove exactly in the ſame manner that, $g\tau = M\tau = cot.\ A$, and $Cg = tang.\ \frac{1}{2} A$; but it is evident that $C\tau = coſec.\ A$; and therefore we ſhall have, $Coſec.\ A = cot.\ A + tang.\ \frac{1}{2} A$. Q. E. 2°. D.

COROLLARY.

20. By comparing the two expreſſions of the *ſecants* and *coſecants* of the arch A M in the two preceding Theorems, it will follow; 1°. that, $cot.\ \frac{1}{2}\ comp.\ A - tang.\ A = tang.\ \frac{1}{2}\ comp.\ A + tang.\ A$; and 2°. that, $cot.\ \frac{1}{2} A - cot.\ A = cot. A + tang.\ \frac{1}{2} A$: conſequently we ſhall likewiſe have, $2\ tang.\ A = cot.\ \frac{1}{2}\ comp.\ A - tang.\ \frac{1}{2}\ comp.\ A$; and, $2\ cot.\ A = cot.\ \frac{1}{2} A - tang.\ \frac{1}{2} A$; and if for the *cot.* in the ſecond member of each of theſe equations there be ſubſtituted its value $\frac{RR}{tang.}$ (*art.* 12.), we ſhall again have, $tang.\ A = \frac{RR - tang.^2\ \frac{1}{2}\ comp.\ A}{2\ tang.\ \frac{1}{2}\ comp.\ A}$; and, $cot.\ A = \frac{RR - tang.^2\ \frac{1}{2} A}{2\ tang.\ \frac{1}{2} A}$; from whence the two following analogies may be eaſily deduced:

1°. As $2\ Tang.\ \frac{1}{2}\ comp.\ A : R + tang.\ \frac{1}{2}\ comp.\ A :: R - tang.\ \frac{1}{2}\ comp.\ A : tang.\ A$; or, which is the ſame, $2\ tang.\ \overline{45^\circ - \frac{1}{2} A} : R + tang.\ \overline{45^\circ - \frac{1}{2} A} :: R - tang.\ \overline{45^\circ - \frac{1}{2} A} : tang.\ A$; and 2°. as $2\ Tang.\ \frac{1}{2} A : R + tang.\ \frac{1}{2} A :: R - tang.\ \frac{1}{2} A : cot.\ A$.

THEOREM

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† *Otherwiſe thus.* Becauſe in the right-angled triangle CMo, the angle CoM is the complement of the angle oCM, or its equal oCB, and likewiſe the angle $oC\theta$ is the complement of the ſame angle oCB to the right angle θCB; therefore the angles θoC and θCo will be equal, and conſequently θC, or $Sec.\ A = \theta o = \theta M + Mo = tang.\ A + tang.\ \frac{1}{2}\ comp.\ A$.

THEOREM VI.

21. *Let there still be an arch* AM *described with a radius* CA; and we shall have (by making the radius unity); 1°. $1 + cos. A = 2\, cos.^2 \frac{1}{2} A = \frac{sin. A \times R}{tang. \frac{1}{2} A}$; Fig. 4.
and 2°. $1 - cos. A = 2\, sin.^2 \frac{1}{2} A = sin. A \times tang. \frac{1}{2} A$.

DEMONSTRATION.

First, let there be drawn through the extremities B and M of the diameter AB and chord AM, the chord BLM, terminated by the tangent of the arch AM at the point T, then let there be drawn through the centre C the right lines CK and CL respectively parallel to the right lines BM and AM; and it is evident that AT will be twice the *tangent*, and CD or ML or LB the *cosine*, of half the arch AM.

This being premised, the similar triangles BLC, BPM, will give, BC : BL : : BM : BP : : 2BL : BP; wherefore, $BP = \frac{2\overline{LB}^2}{BC}$; that is to say, $1 + cos. A = 2\, cos.^2 \frac{1}{2} A$. In like manner, on account of the similar triangles CLB and APM, we shall have, CB : CL : : AM : AP; and therefore $AP = \frac{2\overline{AD}^2}{CB}$, (because AM = 2AD or 2DM or 2CL); that is, $1 - cos. A = 2\, sin.^2 \frac{1}{2} A$. Q. E. 1°. D.

2. From the similar triangles APM, BAT, and BPM, we shall get these two proportions; BP : PM : : BA : AT; or, $1 + cos. A : sin. A : : 2R : 2\, tang \frac{1}{2} A$; and PM : AP : : AB : AT; or, $sin. A : 1 - cos. A : : 2R : 2\, tang. \frac{1}{2} A$: therefore, $1 + cos. A = \frac{sin. A \times R}{tang. \frac{1}{2} A}$; and, $1 - cos. A = \frac{sin. A \times tang. \frac{1}{2} A}{R}$. Q. E. 2°. D.

 COROL-

COROLLARY.

22. It follows from hence, that $\frac{1 - cos.\,A}{1 + cos.\,A} = \frac{2\,sin.^2\,\frac{1}{2}A}{2\,cos.^2\,\frac{1}{2}A} = \frac{tang.^2\,\frac{1}{2}A}{RR}$; and, $\frac{1 + cos.\,A}{1 - cos.\,A} = \frac{cot.^2\,\frac{1}{2}A}{RR}$; and it likewise follows, by putting V and v for the *versed-sines* AP and BP of the arch AM, that $V = \frac{2\,sin.^2\,\frac{1}{2}A}{R}$; and, $v = \frac{2\,cos.^2\,\frac{1}{2}A}{R}$; or, $\frac{VR}{2} = sin^2\,\frac{1}{2}A$; and, $\frac{vR}{2} = cos.^2\,\frac{1}{2}A$.—Moreover, if the arch MG, the complement of AM, be called A, we shall get from the similar triangles AMP, MPB, and AMB, the following formulæ*; $R + sin.\,A = 2\,sin.^2\,\overline{45^\circ + \frac{1}{2}A}$; $R - sin.\,A = 2\,sin.^2\,\overline{45^\circ - \frac{1}{2}A}$, and, $\frac{R + sin.\,A}{R - sin.\,A} = \frac{sin.^2\,\overline{45^\circ + \frac{1}{2}A}}{sin.^2\,\overline{45^\circ - \frac{1}{2}A}}$.

PROBLEM I.

Fig. 5. 23. *Any two arcs* AM *and* AN *being given, to find the sine of their sum and difference.*

SOLUTION.

Let the greater arc be denoted by A, and the less by B: let the *sine* MP of the greater arc AM be produced till it meet the radius CN, which passes through

* The proportions, from whence these formulæ are derived, are, AB : BM : : BM : PB; and, AB : AM : : AM : AP; or, $2R : 2\,cos.\,\overline{45^\circ - \frac{1}{2}A} :: 2\,cos.\,\overline{45^\circ - \frac{1}{2}A} : R + sin.\,A$; and, $2R : 2\,sin.\,\overline{45^\circ - \frac{1}{2}A} :: 2\,sin.\,\overline{45^\circ - \frac{1}{2}A} : R - sin.\,A$. But these formulæ may be otherwise obtained by a simple change of the expressions, $1 + cos.\,A = 2\,cos.^2\,\frac{1}{2}A$, and, $1 - cos.\,A = 2\,sin.^2\,\frac{1}{2}A$; into, $1 + sin.\,A = 2\,cos.^2\,\frac{\overline{90^\circ - A}}{2} = 2\,sin.^2\,\overline{45^\circ + \frac{1}{2}A}$, and, $1 - sin.\,A = 2\,sin.^2\,\overline{45^\circ - \frac{1}{2}A}$.

through the extremity of the leſs arc, in the point R; and perpendicular to the ſame radius let the right line ML, (which will be the *ſine* of the ſum of the arcs AM and AN), be ſuppoſed to be drawn. Then the ſimilar triangles CQN, CPR, will give, CQ : QN : : CP : PR; or, *coſ.* B : *ſin.* B : : *coſ.* A : PR $= \frac{cof.\,A \times fin.\,B}{cof.\,B}$; therefore RM $= fin.\,A + \frac{cof.\,A \times fin.\,B}{cof.\,B}$: moreover, the ſimilar triangles NQC, RLM, will give, NC : QC : : MR : ML; or, R : *coſ.* B : : *ſin.* A $+ \frac{cof.\,A \times fin.\,B}{cof.\,B}$: *ſin.* $\overline{A+B} = \frac{fin.\,A \times cof.\,B + cof.\,A \times fin.\,B}{R}$. Q. E. 1°. I.

2. In order to find the *ſine* of the difference of two arcs; we ſhall now regard MN as the greater, which we have called A, and the arc AN as the leſs, which we ſhall ſtill denote by B. This being ſuppoſed, the ſimilar triangles CQN, CLO, give, CQ : QN : : CL : LO; or, *coſ.* B : *ſin.* B : : *coſ.* A : LO $= \frac{cof.\,A \times fin.\,B}{cof.\,B}$; therefore OM $= fin.\,A - \frac{cof.\,A \times fin.\,B}{cof.\,B}$: moreover, on account of the ſimilar triangles MPO, CQN, we ſhall have, CN : CQ : : OM : PM; or, R : *coſ.* B : : *ſin.* A $- \frac{cof.\,A \times fin.\,B}{cof.\,B}$: *ſin.* $\overline{A-B} = \frac{fin.\,A \times cof.\,B - cof.\,A \times fin.\,B}{R}$. Q. E. 2°. I.

PROBLEM II.

24. *To find the coſine of the ſum and difference of any two arcs* A *and* B. *Fig.* 5.

SOLUTION.

The triangles CQN, MPO, being ſimilar, (ſince the ſides of the one are perpendicular to those

those of the other), we shall have, CQ : QN : : MP : PO; or, *cos.* B : *sin.* B : : *sin.* A : PO $= \frac{sin.\ A \times sin.\ B}{cos.\ B}$; therefore CO $= cos.\ A - \frac{sin.\ A \times sin.\ B}{cos.\ B}$: but on account of the similar triangles CQN, CLO, we likewise have, CN : CQ : : CO : CL; or, R : *cos.* B : : *cos.* A $- \frac{sin.\ A \times sin.\ B}{cos.\ B}$: $cos.\ \overline{A+B} = \frac{cos.\ A \times cos.\ B - sin.\ A \times sin.\ B}{R}$. Q. E. 1°. I.

2. If we now regard the arc MN as A, and the arc AN as B; it is evident that CP will be the *cosine* of their difference. This being granted, the similar triangles CQN, MLR, will give, CQ : QN : : ML : LR; or, *cos.* B : *sin.* B : : *sin.* A : LR $= \frac{sin.\ A \times sin.\ B}{cos.\ B}$; therefore CR, or CL + LR $= cos.\ A + \frac{sin.\ A \times sin.\ B}{cos.\ B}$: moreover, on account of the similar triangles CQN, CPR, we shall have, CN : CQ : : RC : CP; or, R : *cos.* B : : *cos.* A $+ \frac{sin.\ A \times sin.\ B}{cos.\ B}$: $cos.\ \overline{A-B} = \frac{cos.\ A \times cos.\ B + sin.\ A \times sin.\ B}{R}$. Q. E. 2°. I.

Corollary I.

25. From the formulæ which we have investigated in the two last Problems, the following proportions will manifestly arise;

Sin. $\overline{A+B}$: *sin.* $\overline{A-B}$: : *sin.* A × *cos.* B + *cos.* A × *sin.* B : *sin.* A × *cos.* B − *cos.* A × *sin.* B, and *Cos.* $\overline{A+B}$: *cos.* $\overline{A-B}$: : *cos.* A × *cos.* B − *sin.* A × *sin.* B : *cos.* A × *cos.* B + *sin.* A × *sin.* B; and if the two last terms in the first proportion be divided by *cos.* A × *cos.* B, and the two last in the second proportion by *cos.* A × *sin.* B; we shall get, (after substituting for the quotients their equal values in the *tangents* and *cotangents* given in *art.* 11,) these two analogies;

Sin.

$Sin.\ \overline{A+B} : sin.\ \overline{A-B} :: tang.\ A + tang.\ B : tang.\ A - tang.\ B$, and $Cos.\ \overline{A+B} : cos.\ \overline{A-B} :: cot.\ B - tang.\ A : cot.\ B + tang.\ A :: cot.\ A - tang.\ B : cot.\ A + tang.\ B$. ——— It is scarcely necessary to observe, that these proportions might be written under the form of an equation.

Corollary II.

26. Since we have, $cos.\ \overline{A+B} : cos.\ \overline{A-B} :: cos.\ A \times cos.\ B - sin.\ A \times sin.\ B : cos.\ A \times cos.\ B + sin.\ A \times sin.\ B$‡; it follows, that we shall also have by one *subtraction*, $cos.\ \overline{A-B} - cos.\ \overline{A+B} = 2\, sin.\ A \times sin.\ B$; and by one *addition*, $cos.\ \overline{A+B} + cos.\ \overline{A-B} = 2\, cos.\ A \times cos.\ B$.

In like manner from the proportion, $sin.\ \overline{A+B} : sin.\ \overline{A-B} :: sin.\ A \times cos.\ B + cos.\ A \times sin.\ B : sin.\ A \times cos.\ B - cos.\ A \times sin.\ B$‡, we shall find by one *addition* and *subtraction*, that, $sin.\ \overline{A+B} + sin.\ \overline{A-B} = 2\, sin.\ A \times cos.\ B$; and, $sin.\ A+B - sin.\ \overline{A-B} = 2\, cos.\ A \times sin.\ B$; and therefore, if we collect these several expressions and divide them by 2, we shall get

$Sin.\ A \times sin.\ B = \frac{cos.\ \overline{A-B} - cos.\ \overline{A+B}}{2}$; and, $sin.\ A \times cos.\ B = \frac{sin.\ \overline{A+B} + sin.\ \overline{A-B}}{2}$: $Cos.\ A \times cos.\ B = \frac{cos.\ \overline{A+B} + cos.\ \overline{A-B}}{2}$; and, $cos.\ A \times sin.\ B = \frac{sin.\ \overline{A+B} - sin.\ \overline{A-B}}{2}$.

COROL-

‡ Should any difficulty arise with respect to the equations deduced from these analogies, it will immediately vanish upon considering, that as both the antecedents and consequents in each are equal, the sum or difference of the 1st and 2d terms must necessarily be equal to the sum or difference of the 3d and 4th.

COROLLARY III.

27. Hence it again follows, that if we know the *sines* and *cosines* of all the arcs below 30°, we shall be able to get the *sines* and *cosines* of all the arcs above 30° to 60° by subtraction only; and therefore, if by this method we compute the *sines* and *cosines* to 45°, we shall have all the *sines* and *cosines* to 90°; since the *cosine* of an arc below 45° is the *sine* of an arc as much above 45°. But that the truth of this Corollary may be clearly apprehended, it must be observed, that the *sine* of 30° being half the chord of 60° will be equal to $\frac{R}{2}$; and therefore if we make $A = 30^\circ$, $\textit{sin.}\,A \times \textit{sin.}\,B$ will be equal to $\frac{R \times \textit{sin.}\,B}{2} = \frac{\textit{cos.}\,\overline{A-B} - \textit{cos.}\,\overline{A+B}}{2}$; and consequently, $R \times \textit{sin.}\,B = \textit{cos.}\,\overline{30^\circ - B} - \textit{cos.}\,\overline{30^\circ + B}$; or, $\textit{cos.}\,\overline{30^\circ + B} = \textit{cos.}\,\overline{30^\circ - B} - \textit{sin.}\,B$: from whence $\textit{cos.}\,\overline{30^\circ + B}$ will become known, since B is known by the hypothesis, and less than 30°. In like manner, since $\textit{sin.}\,A \times \textit{cos.}\,B = \frac{\textit{sin.}\,\overline{A+B} + \textit{sin.}\,\overline{A-B}}{2}$, we shall find, $\textit{cos.}\,B = \textit{sin.}\,\overline{30^\circ + B} + \textit{sin.}\,\overline{30^\circ - B}$; and, $\textit{sin.}\,\overline{30^\circ + B} = \textit{cos.}\,B - \textit{sin.}\,\overline{30^\circ - B}$: therefore it is manifest that the Tables of *sines* and *cosines* may be easily calculated from this Corollary.

COROLLARY IV.

28. If we suppose the arc B successively equal to A, 2 A, 3A, &c. the *sine* of the arc A to be always denoted by *s*, and its *cosine* by *c*, and the expressions of the *sines* of multiple arcs in *sines* and *cosines* to be represented by $\textit{sin.}\,A$, $\textit{sin.}\,2\,A$, $\textit{sin.}\,3\,A$, $\textit{sin.}\,4\,A$ ——— $\textit{sin.}\,n\,A$, it will be easy to construct the

the following Tables by means of the formula in *Prob.* 1. *Sin.* $\overline{A+B} = \textit{sin.}\ A \times \textit{cos.}\ B + \textit{cos.}\ A \times \textit{sin.}\ B$.

$$\textit{Sin.}\ A = \sqrt{rr-cc}.$$
$$\textit{Sin.}\ 2A = 2c\sqrt{rr-cc}.$$
$$\textit{Sin.}\ 3A = \overline{4cc-1} \times \sqrt{rr-cc}.$$
$$\textit{Sin.}\ 4A = \overline{8c^3-4c} \times \sqrt{rr-cc}.$$
$$\textit{Sin.}\ 5A = \overline{16c^4-12c^2+1} \times \sqrt{rr-cc}.$$
$$\textit{Sin.}\ 6A = \overline{32c^5-32c^3+6c} \times \sqrt{rr-cc}.$$
$$\textit{Sin.}\ 7A = \overline{64c^6-80c^4+24c^2-1} \times \sqrt{rr-cc}.$$
$$\textit{Sin.}\ 8A = \overline{128c^7-192c^5+80c^3-10c} \times \sqrt{rr-cc}.$$

&c.

OR,

$$\dagger\ \textit{Sin.}\ A = s.$$
$$\textit{Sin.}\ 2A = 2s\sqrt{rr-ss}.$$
$$\textit{Sin.}\ 3A = 3s-4s^3.$$
$$\textit{Sin.}\ 4A = \overline{4s-8s^3} \times \sqrt{rr-ss}.$$
$$\textit{Sin.}\ 5A = 5s-20s^3+16s^5.$$
$$\textit{Sin.}\ 6A = \overline{6s-32s^3+32s^5} \times \sqrt{rr-ss}.$$
$$\textit{Sin.}\ 7A = 7s-56s^3+112s^5-64s^7.$$
$$\textit{Sin.}\ 8A = \overline{8s-80s^3+192s^5-128s^7} \times \sqrt{rr-ss}.$$
$$\textit{Sin.}\ 9A = 9s-120s^3+432s^5-576s^7+256s^9.$$

&c.

It is easy to perceive that all the terms of these equations might be rendered homogeneous, by substituting therein the different powers of the radius or *sine* total, which we have not thought necessary to be specified. It is likewise evident that the first Table expresses the *sines* of multiple arcs in

† See *Maseres*'s Trigonometry.

in *cosines*, and the ſecond in *ſines*, of the ſimple arc. —But that theſe equations may have all the generality which 'tis poſſible to give them, we ſhall ſubjoin the following general formulæ for the two Tables; obſerving, that the ſecond Table muſt have two general formulæ to expreſs it, ſince $\sqrt{rr-ss}$ is found in all the terms of the even, but not in thoſe of the odd, ranks. As to the method of finding theſe formulæ, it is deduced from the conſideration of the coefficients and exponents of the ſeveral terms.

A. *General formula for the firſt Table*, n *being any number whatever.*

$$\text{Sin. } nA = \Big(2^{n-1} c^{n-1} - \overline{n-2} \times 2^{n-3} c^{n-3} + \frac{\overline{n-3}\times\overline{n-4}}{1.2}\, 2^{n-5} c^{n-5} - \frac{\overline{n-4}\times\overline{n-5}\times\overline{n-6}}{1.2.3} \times 2^{n-7} c^{n-7}$$
$$+ \frac{\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times\overline{n-8}}{1.2.3.4}\, 2^{n-9} c^{n-9} - \&c.\Big) \times \sqrt{rr-cc}.$$

B. *Firſt general formula for the ſecond Table*, n *being any odd number.*

$$\text{Sin. } nA = ns - \frac{n\times\overline{nn-1}}{1.2.3}\, s^3 + \frac{n\times\overline{nn-1}\times\overline{nn-9}}{1.2.3.4.5}\, s^5 -$$
$$\frac{n\times\overline{nn-1}\times\overline{nn-9}\times\overline{nn-25}}{1.2.3.4.5.6.7}\, s^7 + \&c.$$

C. *Second general formula for the ſame Table*, n *being any even number.*

$$\text{Sin. } nA = \Big(ns - \frac{n\times\overline{nn-4}}{1.2.3}\, s^3 + \frac{n\times\overline{nn-4}\times\overline{nn-16}}{1.2.3.4.5}$$
$$s^5 - \frac{n\times\overline{nn-4}\times\overline{nn-16}\times\overline{nn-36}}{1.2.3.4.5.6.7}\, s^7 + \ldots\ldots + 2^{n-1}$$
$$s^{n-1}\Big) \times \sqrt{rr-ss}.$$

COROL-

COROLLARY V.

29. Moreover, if we ſtill ſuppoſe the arc B ſucceſſively equal to A, 2A, 3A, 4A, &c. &c. &c. and proceed in a ſimilar manner with the formula of the *coſine* of the ſum of two arcs, found in *Prob.* II. we ſhall again form the two following Tables; one of which expreſſes the *coſines* of multiple arcs in *coſines*, and the other in *ſines* of the ſimple arc.

Coſ. A $= c$.
Coſ. 2 A $= 2cc - 1$.
Coſ. 3 A $= 4c^3 - 3c$.
Coſ. 4 A $= 8c^4 - 8c^2 + 1$.
Coſ. 5 A $= 16c^5 - 20c^3 + 5c$.
Coſ. 6 A $= 32c^6 - 48c^4 + 18c^2 - 1$.
Coſ. 7 A $= 64c^7 - 112c^5 + 56c^3 - 7c$.
&c.

O R,

Coſ. A $= \sqrt{rr - ss}$.
Coſ. 2 A $= -2ss + rr$.
Coſ. 3 A $= \overline{-4ss + rr} \times \sqrt{rr - ss}$.
Coſ. 4 A $= 8s^4 - 8s^2 + 1$.
Coſ. 5 A $= \overline{16s^4 - 12s^2 + 1} \times \sqrt{rr - ss}$.
Coſ. 6 A $= -32s^6 + 48s^4 - 18s^2 + 1$.
Coſ. 7 A $= \overline{-64s^6 + 80s^4 - 24s^2 + 1} \times \sqrt{rr - ss}$.
&c.

D. *General formula for the firſt Table,* n *being any number whatever.*

$$r^{n-1} \text{ Coſ. } nA = 2^{n-1} c^n - nr^2\, 2^{n-3}\, c^{n-2} + \frac{n \times \overline{n-3}}{1.2} r^4\, 2^{n-5}\, c^{n-4} - \frac{n \times \overline{n-4} \times \overline{n-5}}{1.2.3} r^6\, 2^{n-7} c^{n-6} + \frac{n \times \overline{n-5} \times \overline{n-6} \times \overline{n-7}}{1.2.3.4} r^8\, 2^{n-9}\, c^{n-8} - \&c.$$

If the arc denoted by A be obtuse, its *cosine* becomes negative, and then all the formulæ wherein the letter c is involved to odd powers, change their signs; as will likewise the terms of the general formula, when n is an odd number.

F. *General formula for the second Table*, n *being any odd number.*

$$\text{Cos. } nA = (\pm 2^{n-1}s^{n-1} \mp \overline{n-2} \times 2^{n-3}s^{n-3} \pm \frac{\overline{n-3}\times\overline{n-4}}{1.2} 2^{n-5}s^{n-5} \mp \&c.) \times \sqrt{rr-ss}.$$

G. *General formula for the same Table, when* n *is any even number.*

$$r^{n-1} \text{ Cos. } nA = \pm 2^{n-1}s^{n} \mp nr^2\, 2^{n-3}s^{n-2} \pm \frac{n\times\overline{n-3}}{1.2} r^4\, 2^{n-5}s^{n-4} \mp \frac{n\times\overline{n-4}\times\overline{n-5}}{1.2.3} r^6\, 2^{n-7}s^{n-6} \pm \frac{n\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}}{1.2.3.4} r^8\, 2^{n-9}s^{n-8} \pm \&c.$$

The higher signs take place in the first formula, when n is one of the odd numbers 1, 5, 9, 13, &c. and the lower signs, when it is one of the numbers 3, 7, 11, &c.

In like manner in the second formula the higher signs prevail, when the numbers are evenly even; and the lower signs when they are evenly odd.

SCHOLIUM.

30. We might now proceed to deduce from the foregoing general formulæ, several useful and important truths concerning the nature of the roots of equations; but fearing by this means to extend our Work to too great a length, we shall refer our readers to the works of Mr. *Euler*, where this matter is fully and satisfactorily handled; and only apply

apply them to the theory of powers of the *ſines* and *coſines*, in two Problems: the firſt of which requires, when "*Any power of a ſine or coſine is given, to find its expreſſion in ſines and coſines of the ſimple arc and its multiples*;" and the ſecond (the bare ſpecification whereof will be ſufficient), "*To expreſs the ſine or coſine of any multiple arc by powers of the ſine or coſine of the ſimple arc.*" The formulæ, which Mr. *Euler* hath given upon this theory, he deduced from thoſe in *art.* 26; but to us it appears, that they are much more eaſily deducible from the preceding formulæ.

PROBLEM III.

31. *To expreſs powers of the ſine and coſine of any arc by ſines and coſines of this arc and its multiples.*

SOLUTION.

This Problem conſiſts of two parts, one for the *ſines*, and the other for the *coſines*; but the formulæ, which we ſhall firſt inveſtigate, ſhall be thoſe which may contain the values of the powers of the *ſines*; and this we ſhall do by means of the two laſt Tables in *Corol.* IV. and V. wherein the letter s was ſuppoſed to denote the *ſine* of the ſimple arc. Now it will appear with a very little conſideration, that the terms of the odd ranks of the ſecond Table in *art.* 28, will give the values of the odd powers of the letter s, and the terms of the even ranks in the ſecond Table of *Corollary* V. thoſe of its even powers; after exterminating ſuch powers as may be found in the laſt terms.——Thus from *ſin.* $A = s$, we get, *ſin.* $A =$ *ſin.* A: alſo from the equation, *coſ.* $2A = -2s^2 + rr$, we get, 2 *ſin.*$^2 A = rr -$ *coſ.* $2A$: again, from the equation, *ſin.* $3A = 3s - 4s^3$, in the ſecond Table of *Corol.* IV.

deduce, $4\,sin.^3 A = 3\,sin.\ A - sin.\ 3\,A$; likewise from, $cos.\ 4\,A = 8\,sin.^4 - 8\,sin.^2 + 1$, in the second Table of *Corol.* V. we shall get, $8\,sin.^4 A = cos.\ 4\,A + 8\,sin.^2 A - 1 = cos.\ 4\,A - 4\,cos.\ 2\,A + 3$, by substituting for $8\,s.^2 A$ its value, $4rr - 4\,cos.\ 2\,A$, as obtained from the above equation, $2\,sin.^2 A = rr - cos.\ 2\,A$: and by pursuing this process, the following Table will be easily formed.

$Sin.\ A = sin.\ A.$

$2\,Sin.^2 A = r^2 - cos.\ 2\,A.$

$4\,Sin.^3 A = 3\,sin.\ A - sin.\ 3\,A.$

$8\,Sin.^4 A = 3 - 4\,cos.\ 2\,A + cos.\ 4\,A.$

$16\,Sin.^5 A = 10\,sin.\ A - 5\,sin.\ 3\,A + sin.\ 5\,A.$

$32\,Sin.^6 A = 10 - 15\,cos.\ 2\,A + 6\,cos.\ 4\,A - cos.\ 6\,A.$

$64\,Sin.^7 A = 35\,sin.\ A - 21\,sin.\ 3\,A + 7\,sin.\ 5\,A - sin.\ 7\,A.$

$128\,Sin.^8 A = 35 - 56\,cos.\ 2\,A + 28\,cos.\ 4\,A - 8\,cos.\ 6\,A + cos.\ 8\,A.$

$256\,Sin.^9 A = 126\,sin.\ A - 84\,sin.\ 3\,A + 36\,sin.\ 5\,A - 9\,sin.\ 7\,A + sin.\ 9\,A.$

&c.

In the above Table it is manifest, that the odd powers of the *sine* of the arc A are all expressed in *sines*, but its even powers in *cosines*, of its multiple arcs. The law of the coefficients is likewise very observable, being evidently the same with that of the coefficients of a binomial raised to each of the powers; except that in the even powers the abstract number, which is not multiplied by a *cosine* of A, is only half the coefficient of the corresponding term in the like power of the binomial. Hence it will appear, that in order to express this Table of powers, we must necessarily have two general formulæ.

H. *First general formula for the powers of sin.* A *when* n *is odd, beginning at the last Terms of the Table.*

$$2^{n-1} Sin.^n A = \pm sin.\ n\,A \pm n\,sin.\ \overline{n-2}\,A \pm \frac{n \times \overline{n-1}}{1.\,2}\, sin.\ \overline{n-4}\,A \mp \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\,2.\,3} \times sin.\ \overline{n-6}\,A \pm \&c. \ldots$$

$$\ldots\ldots \pm \frac{n \times \overline{n-1} \ldots \times \overline{n-2\ \&c.}}{1.\,2.\,3\ \&c.}\, sin.\ A.$$

K. *Se-*

K. *Second general formula for the powers of the same sine, when* n *is any even number.*

$$2^{n-1}\ \textit{Sin.}^{n}\ A = \pm\ \textit{cos.}\ n\ A \mp n\ \textit{cos.}\ \overline{n-2}\ A \pm \frac{n\times\overline{n-1}}{1.\ 2}\ \textit{cos.}\ \overline{n-4}\ A \mp \frac{n\times\overline{n-1}\times\overline{n-2}}{1.\ 2.\ 3}\ \textit{cos.}\ \overline{n-6}\ A\ ..$$

$$\ldots\ldots \mp \tfrac{1}{2}\times\frac{n\times\overline{n-1}\times\overline{n-2}\ \&c.}{1.\ 2.\ 3\ \&c.}\ \textit{cos.}\ 0\ A.$$

Each of the two preceding formulæ hath two signs; and of these the upper must be used, when n is an odd number and equal to $4m+1$; but the lower, when it is equal to $4m-1$; m representing any number whatever.——In like manner, the higher signs must be used in the second formula, when n is equal to $4m$, m denoting any number at pleasure; but the lower signs, when n is equal to $2m$, and m any odd number.

32. Now in the second place, to find a similar formula for the successive powers of the *cosine* of any are, we must apply the first Table of *Corol.* V. exactly in the same manner as we before applied the two last Tables of *Corol.* IV. and V. and we shall by that means easily form the subsequent Table;

$\textit{Cos.}\ A = \textit{cos.}\ A.$

$2\ \textit{Cos.}^2\ A = 1 + \textit{cos.}\ 2\ A.$

$4\ \textit{Cos.}^3\ A = 3\ \textit{cos.}\ A + \textit{cos.}\ 3\ A.$

$8\ \textit{Cos.}^4\ A = 3 + 4\ \textit{cos.}\ 2\ A + \textit{cos.}\ 4\ A.$

$16\ \textit{Cos.}^5\ A = 10\ \textit{cos.}\ A + 5\ \textit{cos.}\ 3\ A + \textit{cos.}\ 5\ A.$

$32\ \textit{Cos.}^6\ A = 10 + 15\ \textit{cos.}\ 2\ A + 6\ \textit{cos.}\ 4\ A + \textit{cos.}\ 6\ A.$

$64\ \textit{Cos.}^7\ A = 35\ \textit{cos.}\ A + 21\ \textit{cos.}\ 3\ A + 7\ \textit{cos.}\ 5\ A + \textit{cos.}\ 7\ A.$

&c.

Then if we regard the last terms of these equations as the first; or which is the same, take them all backwards, we shall readily obtain the following general formula;

L.

L. $2^{n-1}\,Cos^{n}\,A = cos.\,n\,A + n\,cos.\,\overline{n-2}\,A + \frac{n\times\overline{n-1}}{1.\,2}\,cos.\,\overline{n-4}\,A + \frac{n\times\overline{n-1}\times\overline{n-2}}{1.\,2.\,3}\,cos.\,\overline{n-6}\,A \ldots \ldots\ldots + \frac{n\times\overline{n-1}\times\overline{n-2}\ \&c.}{1.\,2.\,3}\,cos.\,\overline{n-n}\,A$, or $cos.\,A$; according as n is an even or odd number.

SCHOLIUM.

33. Though we have already ſolved the converſe of the laſt Problem; yet as it will admit of other different ſolutions, by taking the equations of the Table in *art.* 28 and 29, backwards, and ſeeking the laws of their terms, we ſhall likewiſe give theſe ſolutions.

PROBLEM IV.

34. *To find general formulæ for transforming the ſine or coſine of any multiple of an arc* A, *into powers of the ſine or coſine of the ſimple arc.*

SOLUTION.

This Problem conſiſts of ſeveral parts: for if the *ſine* of a multiple arc be given, it is manifeſt that it may be expreſſed in *ſines* or *coſines*, according as we make uſe of the ſecond or firſt Table in *art.*28: it is likewiſe evident, that in both theſe caſes different formulæ will be requiſite, according as we begin the ſeries by the firſt or laſt terms; and that in the third place, when we begin the ſeries by the laſt terms, the general formulæ will again be different, according as the number n is even or odd. Now if we ſuppoſe the ſeries to begin by the firſt terms, and it be required to expreſs multiple *ſines* in powers of the *coſine* of the ſimple arc, the formula A in *art.* 28 will ſolve the Problem, whether n repreſents an even or odd number; whilſt the formulæ

B and

B and C in the ſame article will give the general values of multiple *ſines* in powers of the *ſine* of the ſimple arc, viz. B when n is any odd number, and C when it is an even one: but the general formulæ for expreſſing multiple *ſines* in powers of the *ſine* of the ſimple arc, when the ſeries are ſuppoſed to begin by the laſt terms, remain ſtill to be obtained; and theſe will manifeſtly be two, one for n when even, and the other for it when odd.

M. *General formula for reducing any multiple ſine into powers of the ſine of the ſimple arc,* n *being an even number.*

$$\text{Sin. } n\,A = \left(\mp 2^{n-1} s^{n-1} \pm \overline{n-2} \times 2^{n-3} s^{n-3} \mp \frac{\overline{n-3} \times \overline{n-4}}{1.\,2}\, 2^{n-5} s^{n-5} \pm \frac{\overline{n-4} \times \overline{n-5} \times \overline{n-6}}{1.\,2.\,3} \times 2^{n-7} s^{n-7} \mp \text{\&c.} \ldots\ldots + ns\right) \sqrt{rr-ss}.$$

N. *General formula for reducing a multiple ſine into powers of the ſine of the ſimple arc,* n *being any odd number.*

$$\text{Sin. } n\,A = \mp 2^{n-1} s^{n} \pm n \times 2^{n-3} s^{n-2} \mp \frac{n \times \overline{n-3}}{1.\,2}\, 2^{n-5} s^{n-4} \pm \frac{n \times \overline{n-4} \times \overline{n-5}}{1.\,2.\,3}\, 2^{n-7} s^{n-6} \mp$$
$$\frac{n \times \overline{n-5} \times \overline{n-6} \times \overline{n-7}}{1.\,2.\,3.\,4}\, 2^{n-9} s^{n-8} \pm$$
$$\frac{n \times \overline{n-6} \times \overline{n-7} \times \overline{n-8} \times \overline{n-9}}{1.\,2.\,3.\,4.\,5}\, 2^{n-11} s^{n-10} \mp \text{\&c.}$$

O. *General formula for reducing a multiple coſine into powers of the coſine of the ſimple arc,* n *being any even number.*

Coſ.

$$Cos.\ n\,A = \pm r^n \mp \frac{n^2}{1.2} r^{n-2} cos.^2 A \pm \frac{n^2 \times \overline{n^2-4}}{1.2.3.4} r^{n-4} cos.^4 A \mp \frac{n^2 \times \overline{n^2-4} \times \overline{n^2-16}}{1.2.3.4.5.6} r^{n-6} cos.^6 A \pm \frac{n^2 \times \overline{n^2-4} \times \overline{n^2-16} \times \overline{n^2-36}}{1.2.3.4.5.6.7.8} r^{n-8} cos.^8 A \mp \&c.$$

P. *General formula for* n, *when any odd number.*

$$Cos.\ n\,A = \pm n r^{n-1} cos.\ A \mp \frac{n \times \overline{n^2-1}}{1.2.3} r^{n-3} cos.^3 A \pm \frac{n \times \overline{n^2-1} \times \overline{n^2-9}}{1.2.3.4.5} r^{n-5} cos.^5 A \mp \frac{n \times \overline{n^2-1} \times \overline{n^2-9} \times \overline{n^2-25}}{1.2.3.4.5.6.7} r^{n-7} cos.^7 A \pm \frac{n \times \overline{n^2-1} \times \overline{n^2-9} \times \overline{n^2-25} \times \overline{n^2-49}}{1.2.3.4.5.6.7.8.9} r^{n-9} cos.^9 A \mp \&c.$$

These two last formulæ are easily deduced from the first Table of *Corol.* V. (*art.* 29), by observing the law of the coefficients for the terms of the even and odd ranks.—In the first formula M of the *sines*, the higher signs must be taken when n is any of the even numbers denoted by $4m$, and m any number at pleasure; but the lower signs, when it is any of the numbers represented by $2m$, and m any odd number.—In like manner, in the second formula N, the higher signs must be used for all the odd numbers denoted by $4m-1$; and the lower for those represented by $4m+1$; m being any number whatever. We may moreover easily perceive, that in the third series O the *cosine* will be positive or negative, according as n is equal to $4m$ or $2m$; m being any number at pleasure in the first case, and any odd one in the second; and that lastly, when n is odd, the *cosine* will be positive or negative, according as n is equal to $4m+1$, or $4m-1$.

Of the uſe of imaginary Factors in the theory of Sines and Coſines.

35. All the ſeries, which we have hitherto given, were obtained by immediate deduction from the two Problems concerning the *ſine* and *coſine* of the ſum and difference of any two arcs, by ſuppoſing theſe arcs to be equal: but there are ſeveral other methods ſtill remaining whereby ſeries of a different nature from the foregoing may be diſcovered; of which we ſhall however only ſubjoin that wherein *imaginary Factors* are concerned; not becauſe we deem expreſſions abſurd in themſelves to be preferable to others, but becauſe they may ſometimes prove peculiarly ſerviceable in the abridgement of calculations, and the diſcovery of important truths.

PROBLEM V.

36. *To find the Factors of the ſum of two ſquares, as* $aa+bb$; *or, which is the ſame, to diſcover how* a *and* b *muſt be combined by multiplication, ſo that their product may be*, $aa+bb$.

SOLUTION.

Let A and B be the indeterminate quantities which ought to affect the roots, a and b, of the ſquares, ſo that their product may be $aa+bb$, and let us ſuppoſe the Factors to be, $a+\mathrm{A}\,b$ and $a\mathrm{B}+b$. Then multiplying theſe two Factors together we ſhall get, $aa\mathrm{B}+ab\,\mathrm{AB}+ab+\mathrm{A}bb=aa+bb$; and if we compare the terms of this equation together, and ſuppoſe $aa\mathrm{B}=aa$, or $\mathrm{A}bb=bb$, we ſhall find B and A to be each $=1$; from whence it is eaſily inferred, that if the quantities A and B are poſſible, A muſt be equal to B, and $\mathrm{AB}=\mathrm{AA}=\mathrm{BB}$: but ſince in the ſum $aa+bb$ the term ab is wanting,

it neceſſarily follows that, $ab+ab$ AB; or, $ab+ab$ AA, or, $ab+ab$ BB, muſt be$=0$; that is to ſay, $AA+1$ or $BB+1=0$; which is impoſſible, ſince we have already ſeen that AA or $BB=1$: therefore the Factors cannot have the form ſuppoſed; and conſequently the aſſumed quantity cannot be reſolved into Factors; or, which is the ſame, is not a product of the quantities a and b in whatever manner they may be combined. However, if we would ſolve the equations, $AA+1=0$ and $BB+1=0$, notwithſtanding the abſurdity they imply, as before obſerved; we ſhall have, AA or $BB=-1$; and by extracting the roots, $A=\pm\sqrt{-1}$, and $B=\pm\sqrt{-1}$. Theſe expreſſions therefore, which Geometers have called imaginary, are not any real quantities, but only ſigns of an abſurd ſuppoſition, wherein we regard that as the product of two quantities which in reality is not: nevertheleſs as theſe ſigns may have their uſe, no juſt reaſon can be alledged why they ſhould not be employed in calculations; and if ſo, the imaginary Factors of $aa+bb$ will be, $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$; or, $b+a\sqrt{-1}$ and $b-a\sqrt{-1}$, or, $-a+b\sqrt{-1}$ and $-a-b\sqrt{-1}$. Q. E. I.

SCHOLIUM.

37. But though we have ſaid above that $aa+bb$ cannot be a product of the quantities a and b, yet it may perhaps be aſked whether it be not poſſible to find this ſum of the ſquares of a and b by ſome other combination, as, $a+b$ and $a-b$. To try this, let each of the quantities $a+b$ and $a-b$ be ſquared, and we ſhall get, $aa+2ab+bb$ and $aa-2ab+bb$; then let theſe ſquares be added together and their ſum divided by 2, and we ſhall by that

that means obtain the propoſed ſum, $aa + bb$; from whence it is manifeſt that $aa + bb$ is not a product of the quantities $a + b$ and $a - b$, but only half the ſum of their ſquares.

We might extend this theory much farther: however the *little*, which we have ſaid upon it, was in our opinion abſolutely neceſſary to be given, ſince the nature of *Imaginaries* hath not been clearly treated of and explained by any Author, at leaſt by any that we know of, notwithſtanding they have been introduced into the *ſines* and *coſines*. Beſides, there are certain peculiar relations found to prevail between curves of the hyperbolical and elliptical kind, and between logarithms and circular arcs, whereof it will be ſcarcely poſſible to form true and preciſe ideas, without calling in the aſſiſtance of what hath been here ſaid upon imaginary expreſſions.

Corollary I.

38. It follows from hence, that if we would find the imaginary Factors of, $sin.^2 A + cos.^2 A = RR$, we ſhall have, $\overline{cos. A + sin. A\sqrt{-1}} \times \overline{cos. A - sin. A\sqrt{-1}} = RR$. In like manner, if we ſuppoſe another arc B, and would make uſe of imaginary Factors, we ſhall find that, $\overline{cos. A + sin. A\sqrt{-1}} \times \overline{cos. B + sin. B\sqrt{-1}} = cos. A \times cos. B - sin. A \times sin. B + \sqrt{-1} \times \overline{cos. A \times sin. B + sin. A \times cos. B} = cos. \overline{A+B} + \sqrt{-1} \times sin. \overline{A+B}$ by *art.* 23 and 24; and that, $\overline{cos. A - sin. A\sqrt{-1}} \times \overline{cos. B - sin. B\sqrt{-1}} = cos. \overline{A+B} - \sqrt{-1} \times sin. \overline{A+B}$. Alſo by a ſimilar calculation we ſhall find that, $\overline{cos. A \pm \sqrt{-1}\, sin. A} \times \overline{cos. B \pm \sqrt{-1}\, sin. B} \times \overline{cos. C \pm \sqrt{-1}\, sin. C} = cos. \overline{A+B+C} \pm \sqrt{-1}\, sin. \overline{A+B+C}$.

COROLLARY II.

39. Hence it again follows, that if we ſuppoſe the arcs A and B to be equal, and take two Factors, we ſhall have, $\overline{cof.\ A \pm fin.\ A\sqrt{-1}}^{2} = cof.\ 2A \pm fin.\ 2A\sqrt{-1}$: if we take three Factors, we ſhall have, $\overline{cof.\ A \pm fin.\ A\sqrt{-1}}^{3} = cof.\ 3A \pm fin.\ 3A\sqrt{-1}$; and conſequently in general, $\overline{cof.\ A \pm fin.\ A\sqrt{-1}}^{n} = cof.\ nA \pm fin.\ nA\sqrt{-1}$.

COROLLARY III.

40. From the laſt equation we ſhall get by tranſpoſition, $fin.\ nA\sqrt{-1} = \overline{cof.\ A + fin.\ A\sqrt{-1}}^{n} - cof.\ nA$; and alſo on account of the ſign $-$, $fin.\ nA\sqrt{-1} = cof.\ nA - \overline{cof.\ A - fin.\ A\sqrt{-1}}^{n}$: then if we add theſe two equations together and divide by $2\sqrt{-1}$, we ſhall get the following expreſſion for the *ſine* of any multiple arc;

$$Sin.\ nA = \frac{\overline{cof.\ A + fin.\ A\sqrt{-1}}^{n} - \overline{cof.\ A - fin.\ A\sqrt{-1}}^{n}}{2\sqrt{-1}}$$

We might likewiſe find that

$$Cof.\ nA = \frac{\overline{cof.\ A + fin.\ A\sqrt{-1}}^{n} + \overline{cof.\ A - fin.\ A\sqrt{-1}}^{n}}{2}$$

COROLLARY IV.

41. Now if we raiſe each of theſe binomials to the power n by the general formula, all the terms affected with *Imaginaries* will be found to deſtroy, and the two following general ſeries remain; one whereof expreſſes the *ſines* of multiple arcs, and the other the *coſines*.

Q. *Firſt*

Q. *First formula.*

$$Sin.\ n\,A = n\,cos.^{n-1} A \times sin.\ A - \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\ 2.\ 3}\, cos.^{n-3} A \times sin.^{3} A + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1.\ 2.\ 3.\ 4.\ 5}\, cos.^{n-5} A \times sin.^{5} A - \&c.$$

R. *Second formula.*

$$Cos.\ n\,A = cos.^{n} A - \frac{n \times \overline{n-1}}{1.\ 2}\, cos.^{n-2} A \times sin.^{2} A + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1.\ 2.\ 3.\ 4}\, cos.^{n-4} A \times sin.^{4} A - \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}}{1.\ 2.\ 3.\ 4.\ 5.\ 6}\, cos.^{n-6} A \times sin.^{6} A + \&c.$$

Corollary V.

42. If we suppose the arc A to be indefinitely small, we shall have, *sin.* A = A, and, *cos.* A = r or 1: therefore that the arc n A may become an assignable quantity, n itself must necessarily become indefinitely great, and consequently the products, $n \times \overline{n-1}$, $n \times \overline{n-1} \times \overline{n-2}$ *&c.* be reduced to the powers, n^2, n^3, *&c.* Hence if we put the arc n A = c, we shall have, (since *sin.* A = A, and A = $\frac{c}{n}$), *sin.*^{2} A = $\frac{c^2}{n^2}$, *sin.*^{3} A = $\frac{c^3}{n^3}$, *&c. &c.* whilst all the powers of *cos.* A will remain equal to unity; by which means the formulæ in the preceding Corollary, will be reduced to the two following ones, whereby it will be easy to express the *sine* and *cosine* of any arc in parts of that arc, or in decimals of the radius; that is, to calculate the *natural sine* and *cosine* of any arc*.

* See "*Exposition du Calcul Astronomique,*" par Mr. de la Lande, page 21.

S. *First*

S. *First formula.*

$$Sin.\, c = c - \frac{c^3}{1.2.3} + \frac{c^5}{1.2.3.4.5} - \frac{c^7}{1.2.3.4.5.6.7} + \frac{c^9}{1.2.3.4.5.6.7.8.9} - \&c.$$

T. *Second formula.*

$$Cos.\, C = 1 - \frac{c^2}{1.2} + \frac{c^4}{1.2.3.4} - \frac{c^6}{1.2.3.4.5.6} + \frac{c^8}{1.2.3.4.5.6.7.8} - \&c.$$

These fomulæ, it is manifest, will give the *sine* and *cosine* so much the sooner, as they converge the faster; that is, as the number of degrees, which the arc denoted by c contains, is the less. ——There are some other series of a similar nature with these, but as they are not immediately connected with the business in hand, we shall go on to

PROBLEM VI.

Fig. 6. 43. *To find the tangent of the sum or difference of two given arcs* A *and* B.

SOLUTION.

Let AM and AN be the given arcs, the *tangents* of which, AG and AL, denote by t and θ; and it is evident that the line NT, drawn perpendicularly to the end of the radius CN, will be the *tangent* of the sum of these arcs; which denote by T: but to find the *tangent* of the difference of two arcs A and B, let the arc MN be regarded as A, and the arc AM as B; and AL will manifestly be the *tangent* of their difference; which call τ. —— This being premised, let fall the perpendiculars, AQ, GP, upon the radius CN:

CN: then on account of the ſimilar triangles, CRN, CAQ, we ſhall have, CR : RN : : CA : AQ; or, $\sqrt{rr+\theta\theta} : \theta : : r : \frac{r\theta}{\sqrt{rr+\theta\theta}}$: likewiſe on account of the ſimilar triangles, AQL, GPL, we ſhall have, AL : AQ : : GL : GP; or, $\theta : \frac{r\theta}{\sqrt{rr+\theta\theta}} : : t+\theta : \frac{\overline{t+\theta}\times r}{\sqrt{rr+\theta\theta}}$: moreover, ſince CL : AL : : AL : QL, we ſhall get, $QL = \frac{\theta\theta}{\sqrt{rr+\theta\theta}}$; and alſo from the ſimilar triangles, AQL, GPL, AL : QL : : GL : PL; or, $\theta : \frac{\theta\theta}{\sqrt{rr+\theta\theta}} : : t+\theta : \frac{\overline{t+\theta}\times\theta}{\sqrt{rr+\theta\theta}}$; and conſequently, CP, or $CL - PL = \sqrt{rr+\theta\theta} - \frac{\overline{t+\theta}\times\theta}{\sqrt{rr+\theta\theta}} = \frac{rr-\theta t}{\sqrt{rr+\theta\theta}}$. Now by means of theſe expreſſions it will be eaſy to find that of the *tangent* NT: for becauſe of the ſimilar triangles, GCP, CNT, we ſhall have, CP : PG : : CN : NT; or algebraically, $\frac{rr-\theta r}{\sqrt{rr+\theta\theta}} : \frac{\overline{t+\theta}\times r}{\sqrt{rr+\theta\theta}} : : r : \frac{rr\times\overline{\theta+t}}{rr-\theta t} = T = tang.\ \overline{A+B}$; whence we deduce, by making the radius equal to unity, *Tang.* $\overline{A+B} = \frac{tang.\ A + tang.\ B}{1 - tang.\ A \times tang.\ B}$. Q. E. 1°. I.

2. In order to obtain the *tangent* of the difference of the arcs A and B, let the laſt equation, $T = \frac{rr\times\overline{\theta+t}}{rr-\theta t}$, be reſumed, and one of the tangents θ or t be conſidered as unknown; and we ſhall get by the common rules of Algebra, t or $\tau = \frac{rr\times\overline{T-\theta}}{rr+\theta T}$; that

that is, by making the radius equal to unity, r or $Tang.\ \overline{A-B} = \frac{tang.\ A - tang.\ B}{1 + tang.\ A \times tang.\ B}$. Q. E. 2°. I.

SCHOLIUM.

44. We might have found the same results without any geometrical construction by means of the formulæ already given, as follows. Let S be put for the secant of A, and s for that of B: then by the known properties of *sines*, *cosines*, *tangents* and *secants*, we shall have; $sin.\ A = \frac{Tr}{S}$, and, $cos.\ A = \frac{rr}{S}$; $sin.\ B = \frac{tr}{s}$, and, $cos.\ B = \frac{rr}{s}$; which values being substituted in those of, $sin.\ \overline{A \pm B}$ and $cos.\ \overline{A \pm B}$, we shall get, $sin.\ \overline{A+B} = \frac{r^2 \times \overline{T+t}}{Ss}$, and, $sin.\ \overline{A-B} = \frac{r^2 \times \overline{T-t}}{Ss}$; $cos.\ \overline{A+B} = \frac{r \times \overline{r^2 - Tt}}{Ss}$, and, $cos.\ \overline{A-B} = \frac{r \times \overline{rr+Tt}}{Ss}$; and therefore, since $tang. = \frac{r \times sin.}{cos.}$, we shall find as before, $tang.\ \overline{A+B} = \frac{r^2 \times \overline{T+t}}{rr - Tt}$, and, $tang.\ \overline{A-B} = \frac{r^2 \times \overline{T-t}}{rr+Tt}$.

COROLLARY I.

45. $Tang.\ \overline{A+B} \times tang.\ \overline{A-B} =$

$$\frac{\overline{tang.\ A + tang.\ B} \times \overline{tang.\ A - tang.\ B} \times r^4}{\overline{rr - tang.\ A \times tang.\ B} \times \overline{rr + tang.\ A \times tang.\ B}} = \frac{\overline{tang.^2 A - tang.^2 B} \times r^4}{r^4 - tang.^2 A \times tang.^2 B}.$$

COROLLARY II.

46. If one of the angles be 45°, we shall have, $tang.\ \overline{A + 45°} = \frac{\overline{1 + tang.\ A} \times r}{1 - tang.\ A} = \frac{rr}{tang.\ \overline{45° - A}}$; because

becaufe the radius is (by *art.* 12) a mean proportional between *tang.* $\overline{45^\circ + A}$ and *tang.* $\overline{45^\circ - A}$: from whence it follows, that if we calculate the *tangents* of all the arcs below 45°, we fhall get the *tangents* of all the arcs above 45°, by fimple divifions; and, if we compute the logarithms of the fame *tangents*, we fhall likewife get the logarithms of the *tangents* of all the arcs above 45°, by fubtracting the logarithms of the *tangents* below 45° from double the logarithm of the fine total.——This Corollary therefore fhews an eafy method of conftructing the Tables of *natural* and *artificial* tangents.

Corollary III.

47. Whilft the arcs A and B continue fuch, that their fum is lefs than 90°, the expreffion of *tang.* $\overline{A+B}$ will be pofitive: if we fuppofe *tang.* A $\times$ *tang.* B to be $= rr$, (which happens when thefe arcs are the complements to each other), the denominator then becomes nothing, and confequently the *tangent* infinite; but when *tang.* A $\times$ *tang.* B is greater than rr, (which is the cafe when the faid arcs taken together conftitute an arc greater than 90°), the expreffion of the *tangent* then becomes negative, and of confequence muft be taken in an oppofite fenfe †.

Corollary IV.

48. If we fuppofe the arcs A and B to be equal, and imagine a feries of arcs, A, 2A, 3A, 4A, &c. multiples of the firft; it will be eafy to form the following Table of their *tangents* from the formu-

† See *Maferes's* Elements of Trigonometry, p. 78, &c. or, *Simpfon's* Trigonometry, p. 56.

la, *Tang.* $\overline{A+B} = \dfrac{r^2 \times \overline{tang.\ A + tang.\ B}}{r^2 - tang.\ A \times tang.\ B}$; by always regarding the *tangent* laſt found as *tang.* A, and making B equal to the ſimple arc A, whereof we ſeek the multiples,

Tang. $A = T$.

Tang. $2A = \dfrac{2r^2 T}{rr - TT}$.

Tang. $3A = \dfrac{3rrT - T^3}{r^2 - 3TT}$.

Tang. $4A = \dfrac{4r^4 T - 4r^2 T^3}{r^4 - 6r^2 T^2 + T^4}$.

Tang. $5A = \dfrac{5r^4 T - 10r^2 T^3 + T^5}{r^4 - 10r^2 T^2 + 5T^4}$.

And from hence it will likewiſe be eaſy to deduce the following general formula, by obſerving the law of the coefficients for any multiple arc, and making the radius equal to unity.

Tang. $nA =$

$$\frac{nT - \dfrac{n \times \overline{n-1} \times \overline{n-2}}{1.\ 2.\ 3} T^3 + \dfrac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1.\ 2.\ 3.\ 4.\ 5} T^5 - \&c.}{1 - \dfrac{n \times \overline{n-1}}{1.\ 2} T^2 + \dfrac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1.\ 2.\ 3.\ 4} T^4 - \&c.}$$

But in order to abridge this formula, (as well as all thoſe which have been hitherto given), we may obſerve, that every ſucceeding coefficient contains that of the term which precedes it; and therefore, if the coefficients of the numerator be repreſented by the indeterminate letters, A, B, C, D, &c. and thoſe of the denominator by, α, β, γ, δ, &c. the foregoing formula may be expreſſed thus;

Tang. $nA =$

$$\frac{nT - A.\dfrac{\overline{n-1}.\overline{n-2}}{2.\ 3} T^3 + B.\dfrac{\overline{n-3}.\overline{n-4}}{4.\ 5} T^5 - C.\dfrac{\overline{n-5}.\overline{n-6}}{6.\ 7} T^7 + \&c.}{1 - \dfrac{n \times \overline{n-1}}{1.\ 2} T^2 + \alpha.\dfrac{\overline{n-2}.\overline{n-3}}{3.\ 4} T^4 - \beta.\dfrac{\overline{n-4}.\overline{n-5}}{5.\ 6} T^6 + \&c.}.$$

49. If

49. If we would expreſs the *tangents* of multiple arcs by beginning at the higheſt terms or higheſt powers of T, we ſhall readily perceive, that two general ſeries or formulæ will be requiſite; the one for odd numbers, and the other for even ones; and that, ſince the higheſt odd powers are alternately poſitive and negative, the multiple *tangent* muſt have the two ſigns $\pm$ prefixed to it.

Firſt formula for n *when odd.*

$$\pm \textit{Tang.}\, n\,\mathrm{A} =$$

$$\frac{T^n - \frac{n.\overline{n-1}}{1.2}T^{n-2} + A.\frac{\overline{n-2}.\overline{n-3}}{3.4}T^{n-4} - B.\frac{\overline{n-4}.\overline{n-5}}{5.6}T^{n-6} + \&c.}{nT^{n-1} - \alpha.\frac{\overline{n-1}.\overline{n-2}}{2.3}T^{n-3} + \beta.\frac{\overline{n-3}.\overline{n-4}}{4.5}T^{n-5} - \gamma.\frac{\overline{n-5}.\overline{n-6}}{6.7}T^{n-7} + \&c.}$$

Second formula for n *when any even number.*

$$\pm \textit{Tang.}\, n\,\mathrm{A} =$$

$$\frac{nT^{n-1} - A\frac{\overline{n-1}.\overline{n-2}}{2.3}T^{n-3} + B.\frac{\overline{n-3}.\overline{n-4}}{4.5}T^{n-5} - \&c.}{T^n - \frac{n.\overline{n-1}}{1.2}T^{n-2} + \alpha.\frac{\overline{n-2}.\overline{n-3}}{3.4}T^{n-4} - \&c.}$$

50. If we would have the *cotangents* concerned in the expreſſions of the *tangents* of multiple arcs, we have nothing more to do, than to ſubſtitute *cot.* in the place of $\frac{RR}{T}$; after which, it will be eaſy to change the preceding Table into the following one for the *cotangents* of multiple arcs.

$Cot.\ A = cot.\ A.$

$$Cot.\ 2A = \frac{cot.\ A - tang.\ A}{2}$$

$$Cot.\ 3A = \frac{r^2\ cot.\ A - 3r^2\ tang.\ A}{3r^2 - tang^2.\ A}$$

$$Cot.\ 4A = \frac{r^2\ cot.\ A - 6r^2\ tang.\ A + tang.^3 A}{4r^2 - 4\ tang^2.\ A}$$

$$Cot.\ 5A = \frac{r^2\ cot.\ A - 10r^2\ tang.\ A + 5\ tang^3.\ A}{5r^2 - 10\ tang^2.\ A + tang^4.\ A}$$

$$Cot.\ 6A = \frac{r^4\ cot.\ A - 15\ r^4 tang.\ A + 15r^2 tang^3.\ A - tang.^5\ A}{6r^4 - 20r^2\ tang^2.\ A + 6\ tang^4.\ A}$$

And in general we shall have;

$Cot.\ nA =$

$$\frac{cot.\ A - \frac{n.\overline{n-1}}{1.\ 2} tang.\ A + A.\frac{\overline{n-2}.\overline{n-3}}{3.\ 4} tang.^3\ A - \&c.}{n - \alpha.\frac{\overline{n-1}.\overline{n-2}}{2.\ 3} tang.^2 A + \beta.\frac{\overline{n-3}.\overline{n-4}}{4.\ 5} tang.^4 A - \&c.}$$

by making the radius equal to unity; because by specifying the radius, two series would be necessary for the *cotangents* of multiple arcs, according as n should be an even or odd number.

51. The two last general formulæ might likewise, it is manifest, be applied to the *cotangents* of multiple arcs, by beginning at the terms of the highest powers, since the coefficients are always the same. We may also make use of them, to transform the powers of *tangents* into *tangents* of the simple arc and its multiples: thus, if we suppose, T′, T″, T‴, &c. to denote the *tangents* of multiple arcs, we shall easily obtain the following Table by simple substitutions.

T=

$$T = T.$$

$$T^2 = \frac{T'' - 2T}{T''}.$$

$$T^3 = \frac{3T''T - 6T'''T + 2T'''T''}{T''}.$$

$$T^4 = \frac{-8TT'' + 24TT'' - 8T''T'' - 12TT^{IV} + 5T''T^{IV}}{T''T^{IV}}.$$

&c.

COROLLARY.

52. It follows from hence, that in order to obtain all the *tangents* from 0° to 90°, it will be ſufficient to find them to 30°: for if any arc leſs than 30° be called B; and we have, $A = 30^\circ - B$; we ſhall alſo have, $2\,A = 60^\circ - 2\,B$, and, *cot.* $2\,A =$ *tang.* $\overline{30^\circ + 2B}$; and therefore, on account of the formula (*art.* 20), *cot.* $2A = \frac{1}{2}$*cot.*$A - \frac{1}{2}$ *tang.* A, we ſhall get, *tang.* $\overline{30^\circ + 2B} = \frac{1}{2}$ *cot.*$\overline{30^\circ - B} - \frac{1}{2}$ *tang.*$\overline{30^\circ - B}$: from whence it eaſily appears, how the calculation of the Tables of *tangents* may be as much ſhortened, as we have already ſhewn that the calculation of thoſe of the *ſines* and *coſines* might be.

FIRST SCHOLIUM.

53. By the formulæ, which we have given in the preceding general ſeries, we may not only find the *ſines*, *coſines*, *tangents* and *cotangents* of multiple arcs, but alſo the *ſine*, *coſine*, *&c.* of any ſubmultiple arc or angle in any relation whatever; but then, it is manifeſt, we ſhall have equations of different dimenſions concerned; which, when n is a whole or commenſurable number, will be always finite, and admit of eaſy ſolutions by means of the Tables of *ſines*, &c. For example, if we would divide an arc into three or five equal parts, by means of its *ſine* or *coſine*, we need only make, *ſin.* $n\,A = a$, and, *coſ.* $n\,A = b$; $s = x$, and, $c = y$; and the formulæ B and

B and D will give the following equations, $4x^3 - 3r^2x + ar^2 = 0$, and, $4y^3 - 3r^2y - br^2 = 0$. In like manner, supposing n still $= 3$; *tang.* n A$=c$, and, *cot.* n A$=d$; *tang.*A$=z$, and, *cot.* A$=u$; the formulæ in *art.* 48 and 50 give, $z^3 - 3cz^2 - 3r^2z + r^2c = 0$, and, $u^3 - 3du^2 - 3r^2u + r^2d = 0$: from whence it follows, that the trisection of an angle always produces an equation of three dimensions, in whatever manner its solution be attempted.

SECOND SCHOLIUM.

Fig. 7. 54. But, that we may be able to form clear ideas of the changes whereof the preceding formulæ are susceptible with regard to the signs; let us conceive a radius CA to remain fixed, whilst a moveable radius CM, by describing the circumference ABabA, gives the several arcs AM, Am, Aμ, &c. and it will appear, by taking the point A for the common origin of all these arcs, as well as of their *tangents*, that all the possible changes relating to the *sines*, *cosines*, *tangents* and *cotangents*, are reduced to the following:

1°. The *sine* of the nascent arc hath 0 for its limit, and its *cosine* equal to radius; therefore, since *tang.* $= \frac{R \times fin.}{cof.}$, we shall have *tang.*$=0$, and *cot.*$=\infty$. 2°. The *sine* increases, and *cosine* decreases, from 0 to 90°; therefore the *tang.* increases, and *cot.* decreases, positively. 3°. At 90°, *sin.*$=$R, and *cos.* $=0$; therefore *tang.* $=\infty$, and *cot.* $=0$. 4°. From this point to 180°, the *sine* always decreases positively, and the *cosine* increases negatively; therefore the *tang.*decreases, and *cot.* increases, negatively. 5°. At 180°, *sin.*$=0$, and *cos.*$=-$R; therefore *tang.* $=0$, and *cot.*$=-\infty$. 6°. After 180°, the *sine* increases, and *cosine* decreases, negatively; therefore the *tang.* increases, and *cot.* decreases, positively.

7°.

7°. At 270°. *sin.* = — R, and *cos.* = 0; therefore *tang.* = — ∞, and *cot.* = 0. 8°. Lastly, after 270°, the *sine* decreases negatively, and the *cosine* increases positively; therefore, the *tang.* decreases and *cot.* increases, negatively to 360°, where the whole becomes as in *art.* 1. ——— As in trigonometrical calculations only right, obtuse, or acute angles are made use of, all the considerations of the signs are reduc'd to inquire in what cases the formulae, which we have given, indicate an obtuse or acute angle: and this is very simple; for according to our suppositions, whenever the expression of a *cosine* or *cotangent* is found to be negative, we are then to regard the angle, to which these signs belong, as obtuse.

Preparation to the following Theorems.

55. Let PE be the arc which we have denoted by A, and PB that which we have called B. From the point B let there be drawn the lines BK and BH respectively perpendicular to the lines DE and CP. Moreover, having drawn the lines BE and BO to the extremities of the chord EO, let fall upon these lines the perpendiculars CT and CR, terminated by the tangent RBT. This construction being perfectly understood, it will appear; 1° that, the arc $BO = A + B$; 2° that, $BE = A - B$; 3° that, $KO = sin.\ A + sin.\ B$; 4° that, $KE = sin.\ A - sin.\ B$; 5° that, $KM = cos.\ B + cos.\ A$; 6° that, $KB = cos.\ B - cos.\ A$; 7° that, $BT = tang.\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$; 8° that, $BR = tang.\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$; 9° that, $AL = sin.\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$; 10° that, $CL = cos.\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$; 11° that, $BF = sin.\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$; 12° that, $CF = cos.\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$; 13° that, $OM = 2\,CF = 2\,cos.\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$; (for the arc $MO = MN + NO$; but Fig. 8.

but, $NO = 180° - A$, and, $MN = B$; therefore $\frac{MO}{2} = \textit{sin.}\ \overline{90° - \frac{1}{2}A + \frac{1}{2}B}$; and consequently, $MO = 2\ \textit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}$); and 14° that, in like manner, $ME = 2\ CL = 2\ \textit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B}$.——— So much being premised, the following Theorems will be obtained and understood without any difficulty.

Theorem VII.

56. *Supposing all things as in the preceding construction, I say, that we shall have*; 1° $\textit{Sin.}\ A + \textit{sin.}\ B = 2\ \textit{sin.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \textit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}$, and 2° $\textit{Sin.}\ A - \textit{sin.}\ B = 2\ \textit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B} \times \textit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B}$.

DEMONSTRATION.

The right-angled triangles, MKO, CLA, are manifestly similar, since the angle at M in the former is equal to that at C in the latter; and therefore we shall have, $OK : OM :: AL : AC$; or, by substituting for these lines their expressions in *sines* and *cosines*, $\textit{Sin.}\ A + \textit{sin.}\ B : 2\ \textit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B} :: \textit{sin.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} : R$; from whence, by making the radius equal to unity, the first part of the Theorem is deduced. Q. E. 1°. D.

2°. On account of the similar triangles, EKB, CLA, we shall also have, $EK : EB :: CL : CA$; or, $\textit{Sin.}\ A - \textit{sin.}\ B : 2\ \textit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B} :: \textit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} : R$; from whence we deduce the second part of the Theorem. Q. E. 2°. D.

Theorem VIII.

57. *I say moreover, that we shall have the two following equations*; 1° $\textit{Cos.}\ A + \textit{cos.}\ B = 2\ \textit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \textit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}$; 2° $\textit{Cos.}\ B - \textit{cos.}\ A = 2\ \textit{sin.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \textit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}$.

DE-

DEMONSTRATION.

The ſame ſimilar triangles, MKO, CLA and EKB, give yet the two following analogies; MK : MO : : CL : CA, and, BK : BE : : AL : CA; which, by ſubſtituting the particular value of each term, will be thus expreſſed: $\mathit{Cos.}\ A + \mathit{cos.}\ B : 2\ \mathit{cos.}\ \overline{\tfrac{1}{2} A - \tfrac{1}{2} B} :: \mathit{cos.}\ \overline{\tfrac{1}{2} A + \tfrac{1}{2} B} : R$, and, $\mathit{Cos.}\ B - \mathit{cos.}\ A : 2\ \mathit{sin.}\ \overline{\tfrac{1}{2} A - \tfrac{1}{2} B} :: \mathit{sin.}\ \overline{\tfrac{1}{2} A + \tfrac{1}{2} B} : R$: from whence are immediately deduced the two equations which were required to be demonſtrated.

COROLLARY I.

58. Hence, it is manifeſt that, we ſhall get, $\frac{\mathit{sin.}\ A + \mathit{sin.}\ B}{\mathit{sin.}\ A - \mathit{sin.}\ B} = \frac{2\ \mathit{sin.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}}{2\ \mathit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}} = \frac{\mathit{tang.}\ \overline{\frac{1}{2}A + \frac{1}{2}B}}{\mathit{tang.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}}$ $= \mathit{tang.}\ \overline{\tfrac{1}{2} A + \tfrac{1}{2} B} \times \mathit{cot.}\ \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}$, by ſubſtituting *tang.* for $\frac{\mathit{sin.}}{\mathit{cos.}}$ &c.

COROLLARY II.

59. Alſo, $\frac{\mathit{sin.}\ A + \mathit{sin.}\ B}{\mathit{cos.}\ A + \mathit{cos.}\ B} = \frac{2\ \mathit{sin.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}}{2\ \mathit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}}$ $= \mathit{tang.}\ \overline{\tfrac{1}{2} A + \tfrac{1}{2} B}$, by expunging the quantities that deſtroy, and putting *tang.* for $\frac{\mathit{sin.}}{\mathit{cos.}}$

COROLLARY III.

60. We ſhall moreover find that, $\frac{\mathit{sin.}\ A - \mathit{sin.}\ B}{\mathit{cos.}\ A + \mathit{cos.}\ B} = \frac{2\ \mathit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}}{2\ \mathit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{cos.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}} = \mathit{tang.}\ \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$; that $\frac{\mathit{sin.}\ A - \mathit{sin.}\ B}{\mathit{cos.}\ B - \mathit{cos.}\ A} = \frac{2\ \mathit{cos.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}}{2\ \mathit{sin.}\ \overline{\frac{1}{2}A + \frac{1}{2}B} \times \mathit{sin.}\ \overline{\frac{1}{2}A - \frac{1}{2}B}} = \mathit{cot.}$

$\overline{\tfrac{1}{2} A + \tfrac{1}{2} B}$, and that, $\frac{cos. B + cos. A}{cos. B - cos. A} =$

$$\frac{2\, cos. \overline{\tfrac{1}{2} A + \tfrac{1}{2} B} \times cos. \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}}{2\, sin. \overline{\tfrac{1}{2} A + \tfrac{1}{2} B} \times sin. \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}} = \frac{cot. \overline{\tfrac{1}{2} A + \tfrac{1}{2} B}}{tang. \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}} = cot.$$

$\overline{\tfrac{1}{2} A + \tfrac{1}{2} B} \times cot. \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}$.

COROLLARY IV.

61. We might likewise find by simple substitutions (though more immediately from what hath been demonstrated in *art.* 13, $R : cos. \tfrac{1}{2} A :: 2\, sin. \tfrac{1}{2} A : sin. A$, by supposing $A = A \pm B$), that, $\frac{sin. \overline{A+B}}{sin. \overline{A-B}}$ $= \frac{sin. \overline{\tfrac{1}{2} A + \tfrac{1}{2} B} \times cos. \overline{\tfrac{1}{2} A + \tfrac{1}{2} B}}{sin. \overline{\tfrac{1}{2} A - \tfrac{1}{2} B} \times cos. \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}}$, and also that, $\frac{cos. \overline{A+B}}{cos. \overline{A-B}}$ $= \frac{R^2 - 2\, sin.^2 \overline{\tfrac{1}{2} A + \tfrac{1}{2} B}}{R^2 - 2\, sin.^2 \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}}$*. Moreover, we might yet deduce from the different similar triangles, which occur in *fig.* 8, a great number of other properties besides those we have here specified; as the formulæ which are given in *art.* 26, for instance; but what we have already shewn will, we presume, be quite sufficient for pointing out the method of obtaining such properties as we may at any time chance to have occasion for; and therefore it only now remains, that we make some applications of the preceding formulæ to the logarithms.

* Since $R : cos. \tfrac{1}{2} A :: 2\, sin. \tfrac{1}{2} A : sin. A$, we shall have, $R^2 : cos.^2 \tfrac{1}{2} A$ or $R^2 - sin.^2 \tfrac{1}{2} A :: 4\, sin.^2 \tfrac{1}{2} A : sin.^2 A$ or $R^2 - cos.^2 A$; and therefore, $R^4 - R^2 \times cos.^2 A = 4 R^2 \times sin.^2 \tfrac{1}{2} A - 4\, sin.^4 \tfrac{1}{2} A$, or, $R^2 \times cos.^2 A = R^4 - 4 R^2 \times sin.^2 \tfrac{1}{2} A + 4\, sin.^4 \tfrac{1}{2} A$; whence, $R \times cos. A = R^2 - 2\, sin.^2 \tfrac{1}{2} A$, or, $R \times cos. \overline{A \pm B} = R^2 - 2\, sin.^2 \overline{\tfrac{1}{2} A \pm \tfrac{1}{2} B}$, and consequently, $\frac{cos. \overline{A+B}}{cos. \overline{A-B}} = \frac{R^2 - 2\, sin.^2 \overline{\tfrac{1}{2} A + \tfrac{1}{2} B}}{R^2 - 2\, sin.^2 \overline{\tfrac{1}{2} A - \tfrac{1}{2} B}}$.

Use of some of the preceding Formulæ in the Logarithms.

62. The formulæ, $\textit{sin.}\ A+\textit{sin.}\ B=2\,\textit{sin.}\ \overline{\tfrac{1}{2}A+\tfrac{1}{2}B}\times\textit{cos.}\ \overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ and $\textit{sin.}\ A-\textit{sin.}\ B=2\,\textit{cos.}\ \overline{\tfrac{1}{2}A+\tfrac{1}{2}B}\times\textit{sin.}\ \overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ furnish us with the means of obtaining the logarithms of the sum and difference of two given quantities.——An example or two will convey a sufficiently clear idea of the method of operation.

EXAMPLE I.

63. Let the numbers 8467 and 8635 be proposed, and let it be required to find, from the formula, $\textit{sin.}\ A+\textit{sin.}\ B=2\,\textit{sin.}\ \overline{\tfrac{1}{2}A+\tfrac{1}{2}B}\times\textit{cos.}\ \overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$, the logarithm of their sum; which it is imagined cannot be found amongst those of the Tables. (The process would be exactly similar with respect to any two other numbers, even though they should contain fractions, provided their particular logarithms were given.) Now the first thing to be performed, in this example, is to obtain the values of the angles A and B; and to do this, I add 6 to each of the characteristics of the proposed numbers (that I may be able to find the said angles in the Tables of the logarithms of the *sines*, because these logarithms have no characteristic below 7 or above 9), and I thereby get

$9.936262 = \textit{log.}\, 8635 = \textit{log. sin.}\ 59^\circ\ 42'\ 42'' = A.$
$9.927730 = \textit{log.}\, 8467 = \textit{log. sin.}\ 57^\circ\ 51'\ 16'' = B;$
and of consequence deduce, $\tfrac{1}{2}A+\tfrac{1}{2}B=58^\circ\ 46'\ 59''$, and $\tfrac{1}{2}A-\tfrac{1}{2}B=0^\circ\ 55'\ 43''$.

This

This done, I, in the next place, seek the logarithms of the *sine* of the first of the last angles, and that of the *cosine* of the latter of them,

$$\begin{array}{l} 9.932073 = log. sin. \overline{\tfrac{1}{2}A + \tfrac{1}{2}B}. \\ 9.999943 = log. cos. \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}. \\ \underline{0.301030 = log.\ 2.} \\ 20.233046 \\ 4.233046 = log.\ 17102. \end{array}$$

and to the sum of these logarithms add the logarithm of the number 2; then from this sum I subtract 10 + 6, or 16 (on account of the radius which divides, and the six units that were before added to the characteristics), and find that the logarithm of 17102 is equal to the remainder, 4.233046.

Example II.

64. The same formulæ might likewise be made use of in order to construct by the logarithms an expression more complicated than the preceding; such as, for instance, this, $r \times \frac{\overline{AA + aa - bb}}{2Aa}$: wherein I shall suppose r to represent the sine total; that A is a number whose logarithm $= 4.054723$; that the logarithm of $a = 3.108354$, and that of $b = 3.876870$; and consequently that the logarithms of the quantities, A^2, a^2 and b^2 = the doubles of these. Thus much being supposed, I first regard the logarithms of A^2 and a^2 as those of the *sines* of two angles, whose sum is to be found: then I add unity to each of their characteristics (that I may be enabled to find their logarithms amongst those of the *sines*), and, having found the angles corresponding to the resulting sums, perform the operation

tion in the manner annexed; and subtracting 10 1 from the characteristic of the number last found, get the lo-

$$\begin{array}{l} log.\, A^2 = 9.109446 = log.\, sin.\, 7^\circ\ 23'\ 32'' = A. \\ log.\, a^2 = 7.216708 = log.\, sin.\, 0^\circ\ 5'\ 40'' = B. \\ \quad \textit{therefore}\ \tfrac{1}{2}A + \tfrac{1}{2}B = 3^\circ\ 44'\ 36''. \\ \quad \textit{and}\ \tfrac{1}{2}A - \tfrac{1}{2}B = 3^\circ\ 38'\ 56''. \end{array}$$

$$\begin{array}{l} 0.301030 = log.\, 2. \\ 8.814825 = log.\, sin.\, \overline{\tfrac{1}{2}A + \tfrac{1}{2}B}. \\ \underline{9.999119} = log.\, cos.\, \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}. \\ 19.114974 \end{array}$$

garithm of $A^2 + a^2 = 8.114974$: then, again considering this logarithm as that of *sin.* A, and adding unity thereto as well as to the logarithm of b^2, I complete the operation as follows, by means of the formula, $sin.\, A - sin.\, B = 2\, sin.\, \overline{\tfrac{1}{2}A - \tfrac{1}{2}B} \times cos.\, \overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$;

$$\begin{array}{l} 9.114974 = log.\, sin.\, 7^\circ\ 29'\ 14''. \\ 8.753740 = log.\, sin.\, 3^\circ\ 15'\ 6''. \end{array} \quad \textit{therefore} \begin{array}{l} \tfrac{1}{2}A + \tfrac{1}{2}B = 5^\circ\ 22'\ 10''. \\ \tfrac{1}{2}A - \tfrac{1}{2}B = 2^\circ\ 7'\ 4''. \end{array}$$

$$\begin{array}{l} 0.301030 = log.\, 2. \\ 8.567658 = log.\, sin.\, \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}. \\ \underline{9.998090} = log.\, cos.\, \overline{\tfrac{1}{2}A + \tfrac{1}{2}B}. \\ 18.866778 \end{array} \qquad \begin{array}{l} 0.301030 = log.\, 2. \\ 4.054723 = log.\, A. \\ \underline{3.108354} = log.\, a. \\ 7.464107 = log.\, 2\, A\, a. \end{array}$$

$$7.866778 = log.\, A^2 + a^2 - b^2.$$

$$\begin{array}{l} 17.866778 = log.\, \overline{A^2 + a^2 - b^2} \times r. \\ \underline{7.464107} = log.\, 2\, A\, a. \\ \underline{10.402671} = log.\, \dfrac{A^2 + a^2 - b^2}{2\, A\, a} \times r. \end{array}$$

Of the Use of the arithmetical Complement.

65. Before I put an end to these observations, it will not be improper to ascertain the meaning and give a precise notion of what is called the *arithmetical Complement*, which Geometers very frequently use to convert

convert ſubtraction into addition.—And that this may be done, let it be required to ſubtract 754 from 896. Now I eaſily perceive, that if I take the ſubtrahend, 754, from 1000, add the difference to 896, and from the ſum thence ariſing take the figure which is found in the place of thouſands, the ſame remainder will be left as if I ſubtract it from 896, by the common method: but, in order to ſubtract this number, 754, from 1000, it will be ſufficient to put down ſuch figures as make 9 with each of thoſe of the ſubtrahend, the laſt figure excepted, which muſt always make 10, as is proved and demonſtrated by ſubtraction; and the number reſulting from this operation, that is to ſay, (in the preſent caſe) 246 is the *arithmetical Complement*: conſequently, if I add this number to 896, and afterwards from their ſum, 1142, take the unit which poſſeſſes the place of thouſands, I ſhall get 142, for the true remainder ſought.

COROLLARY.

66. From the preceding reaſoning it follows, that the logarithms of the formulæ, $cot.\ A = \frac{RR}{tang.\ A}$; $tang.\ A = \frac{RR}{cot.\ A}$; $coſec.\ A = \frac{RR}{ſin.\ A}$ &c. will be found by adding the arithmetical Complements of the denominators to the logarithm of the radius; or, which is the ſame, by ſubtracting the logarithms of the denominators from the logarithm of the ſquare of the radius, which hath 20 for its index.

SCHOLIUM.

67. As it may be ſometimes neceſſary to make different combinations or ſubſtitutions of the formulæ which we have demonſtrated in this Chapter, and

and as it would frequently prove inconvenient to be obliged to seek for them in the different articles wherein they are specified, we hope that our readers will not be displeased to find them all collected together and united under one view; particularly, as such an arrangement will serve to refer to, whenever, in the following parts of this work, we shall have any substitution to make. We have taken the greatest care possible to preserve throughout the homogeneity of the terms, that we may be enabled to apply these formulæ, as occasion shall require, with the utmost precision and the least danger of mistake.

A general Table of the Formulæ demonstrated in this Chapter.

68. $Sin.\ \frac{1}{2}\ A = cos.\ \frac{1}{2}\ sup.\ A : tang.\ \frac{1}{2}\ A = cot.\ \frac{1}{2}\ sup.\ A$ (*art.* 7).

69. $\frac{Sin.\ A \times R}{cos.\ A} = tang.\ A$ (*art.* 9) $= \frac{sin.\ A \times sec.\ A}{R}$ (*art.* 15).

70. $\frac{Cos.\ A \times R}{sin.\ A} = cot.\ A$ (*art.* 10) $= \frac{RR}{tang.\ A}$ (*art.* 12).

71. $Sin.\ 2\ A = \frac{cos.\ A \times 2\ sin.\ A}{R}$ (*art.* 13); and, $cos.\ 2\ A = \frac{2\ cos.^2\ A - RR}{R}$.

72. $\frac{Sin.\ 2\ A}{2\ sin.\ A} = \frac{cos.\ A}{R} = \frac{RR - 2\ sin.^2\ \frac{1}{2}\ A}{RR}$ (*art.* 21).

73. $\frac{1}{2}\ Sin.\ 2A = \frac{cos.\ A \times sin.\ A}{R} = \frac{sin.^2\ A \times cot.\ A}{RR} = \frac{sin.^2\ A}{tang.\ A}$ (*art.* 14).

74. $R + cos.\ A = \frac{2\ cos.^2\ \frac{1}{2}\ A}{R} = \frac{sin.\ A \times R}{tang.\ \frac{1}{2}\ A}$ (*art.* 21) $= sin.v.\ A$ (*art.* 22) : $R + sin.\ A = 2\ sin.^2\ \overline{45^\circ + \frac{1}{2}A}$.

75. R

75. $R - cos.\, A = \frac{2\,sin.^2\, \frac{1}{2} A}{R} = \frac{sin.\, A \times tang.\, \frac{1}{2} A}{R}$ (*art.* 21.) $= sin.\, V.\, A$ (*art.* 22): $R - sin.\, A = 2\,sin.^2\, \overline{45^\circ - \frac{1}{2} A}$.

76. $\frac{R + cos.A}{R - cos.A} = \frac{cot.^2\, \frac{1}{2} A}{RR} : \frac{R - cos.A}{R + cos.A} = \frac{tang^2\, \frac{1}{2} A}{RR}$ (*art.* 22).

77. $\frac{R + sin.\, A}{R - sin.\, A} = \frac{sin.^2\, \overline{45^\circ + \frac{1}{2} A}}{sin.^2\, \overline{45^\circ - \frac{1}{2} A}}$ (d°).

78. $Tang.\, A = \frac{RR - tang.^2\, \frac{1}{2}\, comp.\, A}{2\, tang.\, \frac{1}{2}\, comp.\, A}$ (*art.* 20).

79. $Cot.\, A = \frac{RR - tang.^2\, \frac{1}{2} A}{2\, tang.\, \frac{1}{2} A}$ (d°).

80. $Sec.\, A = \frac{RR}{cos.\, A} = \frac{tang.\, A \times R}{sin.\, A.}$ (*art.* 15 and 69).

81. $Cosec.\, A = \frac{RR}{sin.\, A}$ (*art.* 15).

82. $\frac{Sec.\, A \times R}{cosec.\, A} = \frac{sin.\, A \times R}{cos.\, A} = tang.\, A$ (*art.* 16).

83. $Sec.\, A = cot.\, \frac{1}{2}\, comp.\, A - tang.\, A$ (*art.* 18) $= tang.\, A + tang.\, \frac{1}{2}\, comp.\, A$ (*art* 19) $= \frac{cot.\overline{45^\circ - \frac{1}{2} A} + tang.\overline{45^\circ - \frac{1}{2} A}}{2} = \frac{tang.\overline{45^\circ + \frac{1}{2} A} + tang.\overline{45^\circ - \frac{1}{2} A}}{2}$

84. $Cosec.\, A = cot.\, \frac{1}{2} A - cot.\, A$ (*art.* 18) $= cot.\, A + tang.\, \frac{1}{2} A$ (*art.* 19) $= \frac{cot.\, \frac{1}{2} A + tang.\, \frac{1}{2} A}{2}$.

85. $Sin.\overline{A \pm B} = \frac{sin.A \times cos.B \pm cos.A \times sin.B}{R}$ (*art.* 23).

86. $Cos.\overline{A \pm B} = \frac{cos.A \times cos.B \mp sin.A \times sin.B}{R}$ (*art.* 24).

87. $\frac{Sin.\, \overline{A + B}}{Sin.\, \overline{A - B}} = \frac{tang.\, A + tang.\, B}{tang.\, A - tang.\, B}$ (*art.* 25) $= \frac{2\, sin.\, \overline{\frac{1}{2} A + \frac{1}{2} B} \times cos.\, \overline{\frac{1}{2} A + \frac{1}{2} B}}{2\, sin.\, \overline{\frac{1}{2} A - \frac{1}{2} B} \times cos.\, \overline{\frac{1}{2} A - \frac{1}{2} B}}$ (*art.* 61).

88. $\frac{Cos.\overline{A+B}}{Cos.\overline{A-B}} = \frac{cot. B - tang. A}{cot. B + tang. A}$ (*art.* 25) $=$

$\frac{\overline{R+\sqrt{2}\, sin.\overline{\frac{1}{2}A+\frac{1}{2}B}} \times \overline{R-\sqrt{2}\, sin.\overline{\frac{1}{2}A+\frac{1}{2}B}}}{\overline{R+\sqrt{2}\, sin.\overline{\frac{1}{2}A-\frac{1}{2}B}} \times \overline{R-\sqrt{2}\, sin.\overline{\frac{1}{2}A-\frac{1}{2}B}}} =$

$\frac{R^2 - 2\, sin.^2\overline{\frac{1}{2}A+\frac{1}{2}B}}{R - 2\, sin.^2\overline{\frac{1}{2}A-\frac{1}{2}B}}$ (*art.* 61).

89. $Sin.\ \overline{30^\circ+A} = cos.\ A - sin.\overline{30^\circ-A}$ (*art.* 27).

90. $Cos.\ \overline{30^\circ+A} = cos.\overline{30^\circ-A} - sin. A$ (d°).

91. $Sin.\ A + sin.\ B = 2\, sin.\ \overline{\frac{1}{2}A+\frac{1}{2}B} \times cos.\ \overline{\frac{1}{2}A-\frac{1}{2}B}$ (*art.* 56).

92. $Sin.\ A - sin.\ B = 2\, sin.\ \overline{\frac{1}{2}A-\frac{1}{2}B} \times cos.\overline{\frac{1}{2}A+\frac{1}{2}B}$ (d°).

93. $Cos.\ A + cos.\ B = 2\, cos.\ \overline{\frac{1}{2}A+\frac{1}{2}B} \times cos.\overline{\frac{1}{2}A-\frac{1}{2}B}$ (*art.* 57).

94. $Cos.\ B - cos.\ A = 2\, sin.\ \overline{\frac{1}{2}A+\frac{1}{2}B} \times sin.\ \overline{\frac{1}{2}A-\frac{1}{2}B}$ (d°).

95. $\frac{Sin. A + sin. B}{Sin. A - sin. B} = \frac{tang.\overline{\frac{1}{2}A+\frac{1}{2}B}}{tang.\overline{\frac{1}{2}A-\frac{1}{2}B}}$ (*art.* 58).

96. $\frac{Sin.\ A + sin.\ B}{Cos.\ A + cos. B} = tang.\ \overline{\frac{1}{2}A+\frac{1}{2}B}$ (*art.* 59).

97. $\frac{Sin.\ A + sin.\ B}{Cos.\ B - cos.\ A} = cot.\ \overline{\frac{1}{2}A-\frac{1}{2}B}$ (*art.* 56 and 57).

98. $\frac{Sin.\ A - sin.\ B}{Cos.\ B + cos.\ A} = tang.\ \overline{\frac{1}{2}A-\frac{1}{2}B}$ (*art.* 60).

99. $\frac{Sin.\ A - sin.\ B}{Cos.\ B - cos.\ A} = cot.\ \overline{\frac{1}{2}A+\frac{1}{2}B}$ (d°).

100. $\frac{Cos.\ B + cos.\ A}{Cos.\ B - cos.\ A} = \frac{cot.\overline{\frac{1}{2}A+\frac{1}{2}B}}{tang.\overline{\frac{1}{2}A-\frac{1}{2}B}}$ (d°) $= \frac{sec.A + sec.B}{sec.A - sec.B}$.

101. $Sin.\ A \times sin.\ B = \frac{1}{2} cos.\ \overline{A-B} - \frac{1}{2} cos.\ \overline{A+B}$ (*art.* 26).

102. $Sin.\ A \times cos.\ B = \frac{1}{2} sin.\ \overline{A+B} + \frac{1}{2} sin.\ \overline{A-B}$ (*art.* 26).

103. $Cos.\ A \times sin.\ B = \frac{1}{2} sin.\ \overline{A+B} - \frac{1}{2} sin.\ \overline{A-B}$ (d°).

104. $Cos.\ A \times cos.\ B = \frac{1}{2} cos.\ \overline{A+B} + \frac{1}{2} cos.\ \overline{A-B}$ (d°).

105. $Tang.\ \overline{A+B} = \frac{\overline{tang.\ A + tang.\ B} \times R^2}{R^2 - tang.\ A \times tang.\ B}$ (*art.* 44).

106. $Tang.\ \overline{A-B} = \frac{\overline{tang.\ A - tang.\ B} \times R^2}{R^2 + tang.\ A \times tang.\ B}$ (d°).

107. $Tang.\ \overline{A+45^\circ} = \frac{\overline{R + tang.\ A} \times R}{R - tang.\ A} = \frac{R^2}{tang.\ \overline{45^\circ - A}}$ (*art.* 46).

108. If A exceeds $45^\circ.\ \frac{Tang.\ A+R}{Tang.\ A-R} = \frac{R^2}{tang.\ \overline{A-45^\circ}}$ (d°).

CHAP.

CHAP. II.

Containing the general Properties of right or oblique-angled ſpherical Triangles, and their Reſolution by analogies.

SECTION I.

Of ſpherical Triangles in general.

DEFINITIONS.

109. ANY portion of a ſpherical ſurface, bounded by three arcs of great circles, is called a *ſpherical Triangle*. *Fig.* 9.

COROLLARY.

110. Hence it follows, that the ſmall circles of the ſphere do not fall under the conſideration of ſpherical Trigonometry, ſince ſuch only are uſed therein as have the ſame centre with the ſphere itſelf.

111. Every ſpherical, as well as plane triangle hath eſſentially three ſides and three angles: and if any three of theſe ſix parts be given, by the rules of Trigonometry, the reſt may be found.

SCHOLIUM.

112. In plane Trigonometry, the knowledge of the three angles is not ſufficient for obtaining the three ſides; for, in this caſe, the relations only of the three ſides can be had, and not their abſolute values; whereas, in ſpherical Trigonometry, where the ſides are circular arcs, whoſe values depend on the number of degrees they contain, when the three angles are given, the ſides will alſo become

known.——But there is yet another more remarkable difference between plane and spherical Trigonometry; which is, that in the former, two angles always determine the third; whereas in the latter they never do: and therefore it follows, that the definition as above stated is in strictness applicable only to spherical Trigonometry; as will more clearly appear from the sequel.

113. The sides of a spherical triangle are all arcs of great circles, which, by their intersection upon the surface of the sphere, constitute the said triangle.

114. The angle, which is contained between the arcs of two great circles cutting each other upon the surface of the sphere, (and already defined to be the sides of a spherical triangle), is called a spherical angle; the measure whereof is known from the Elements of Geometry to be equal to that, which is formed by two lines issuing from the same point of, and perpendicular to, the common section of the planes which determine the containing sides.

COROLLARY.

Fig. 9 & 10 115. Hence it follows, that the surface of a spherical triangle, BAC, and the three planes which determine it, form a kind of triangular Pyramid, BGCA, whereof the vertex, G, is at the centre of the sphere, the base a portion of the spherical surface, whilst its faces, AGC, AGB, and BGC, are parts of great circles or circular sectors, and at the same time form the sides of the triangle, BAC.

116. A line, as PG*p*, perpendicular to the plane of a great circle, passing through the centre of the sphere, and terminated by two diametrically opposite points at its surface, is called the *axis* of such circle; and the points, P, *p*, where the *axis* meets the surface, are called the *poles* thereof.

of Moreover, if we conceive an infinite number of leſſer circles, parallel to the ſaid great circle, this *axis* will be perpendicular to them likewiſe, and the points P and *p* be their poles alſo.

COROLLARY I.

117. Hence, each pole of a great circle is 90° diſtant from every point in its circumference, and all the arcs drawn from the pole of a little circle to its circumference are equal to each other.

COROLLARY II.

118. It likewiſe follows from hence, that all the arcs of great circles drawn through the poles of another great circle are perpendicular thereto; for, ſince they are great circles by the ſuppoſition, they all paſs through the centre of the ſphere, and conſequently through the axis of the ſaid circle. The ſame thing may be affirmed with reſpect to ſmall circles.

COROLLARY III.

119. Therefore, in order to find the *poles* of any circle, we need only deſcribe upon the ſurface of the ſphere two great circles perpendicular to its plane; and the points, where theſe circles interſect each other, will be the poles required.

COROLLARY IV.

120. It moreover follows from hence, that if from any point taken upon the ſurface of the ſphere we would draw an arch of a circle, which may meaſure the ſhorteſt diſtance from this point to the circumference of any given circle, we muſt ſo deſcribe this arch, that its prolongation may paſs

through the poles of the ſaid circle: and contrarily, if an arc paſs through the poles of a given circle, it will meaſure the ſhorteſt diſtance from any aſſumed point to the circumference thereof.

Corollary V.

Fig. 10. 121. If upon the ſides, AC and BC, of a ſpherical triangle, BCA, we take the arcs C L and C K, each 90°, and through the radii, GL and GK, draw the circular plane LGK, it is manifeſt, that the point C will be the pole thereof; and as the lines, GK, GL, are both perpendicular to the common ſection of the planes, AGC and BGC, they meaſure by their inclination the angle of theſe planes, and of conſequence the ſpherical angle BCA likewiſe.

Corollary VI.

122. It is alſo manifeſt, that every arc of a leſſer circle deſcribed from the pole C as centre, and containing the ſame number of degrees with the arc KL, is equally proper for meaſuring the angle ACB; though only arcs of great circles are uſed for this purpoſe.

Corollary VII.

123. Therefore, if a ſpherical angle be right, the arcs of the great circles which form it, paſs mutually through the poles of each other; and if the planes of two great circles contain their reſpective axes, or paſs through the poles of each other, the angle which they comprehend is a right one.

Theorem I.

124. *Any two ſides of a ſpherical triangle*, BAC, *are greater than the third.*

DE-

DEMONSTRATION.

This propoſition is a neceſſary conſequence of this; that the ſhorteſt diſtance between any two points, taken upon the ſurface of the ſphere, is the arc of a great circle paſſing through theſe points. Q. E. D.

Theorem II.

125. *The ſum of the three ſides of any ſpherical triangle is leſs than 360°.*

DEMONSTRATION.

Let the ſides, AC and BC, containing any angle A be produced till they meet again in D, and the arcs, DAC, DBC, will be each 180°, ſince all great circles cut each other into two equal parts: we ſhall therefore have, DAC+DBC=360°: but, by the laſt Theorem, DA and DB are greater than AB, and conſequently the three ſides, AB, AC, and BC, when taken together, leſs than 360°. Q. E. D. *Fig.* 10.

Theorem III.

126. *The ſum of the three angles of any ſpherical triangle is greater than two right angles, and leſs than ſix.*

DEMONSTRATION.

If the ſides, AB, AC, and BC, of the ſpherical triangle, ABC, be ſuppoſed indefinitely ſmall, the interſections, formed between the planes which determine theſe ſides, will approach indefinitely near to right lines, and the ſpherical ſurface indefinitely near to a plane ſurface: therefore, the ſaid triangle may in this caſe be conſidered as a plane triangle; but it is well known, that the angles of ſuch

ſuch a triangle are equal to two right ones only; and conſequently, whilſt the ſides of the ſpherical triangle, ABC, are of a finite magnitude, the ſum of its angles will be always greater than two right angles. Q. E. 1°. D.

2. It will readily appear from an inſpection of the 10th *fig.* that each angle of the ſpherical triangle ABC may be obtuſe; but at the ſame time ſuch, that the arc which is the meaſure thereof may be leſs than 180°; ſince then the angle vaniſhes: therefore, if we ſuppoſe all the angles of the ſaid triangle obtuſe, their ſum cannot poſſibly be ever equal to ſix right ones. Q. E. 2°. D.

Corollary I.

127. Hence it follows, that a ſpherical triangle may have its three angles either right or obtuſe; and therefore, the knowledge of any two angles is not ſufficient for diſcovering the value of the third

Corollary II.

128. If the three angles of a ſpherical triangle be right or obtuſe, the three ſides are likewiſe equal to, or greater than, 90°; and if each of the angles be acute, each of the ſides is alſo leſs than 90°; and contrarily.

SCHOLIUM.

129. From theſe Theorems we may perceive, what the difference between plane and ſpherical triangles in reality is. But if, in ſome caſes, their difference is ſo great, yet, on the other hand, there are ſeveral properties, which are common to ſpherical with plane triangles, and which are demonſtrated exactly in the ſame manner. Thus, for inſtance, it might be here proved,

(as

(as well as in elementary Geometry), that two ſpherical triangles are equal to each other: 1°, when the three ſides of the one are equal to the three ſides of the other; 2°, when they have an equal angle contained between equal ſides, and 3°, when they have equal angles above equal baſes. We might likewiſe ſhew, that a ſpherical triangle is equilateral, iſoſceles or ſcalenous, according as it hath three equal, two equal, or three unequal, angles; and contrarily; and laſtly, that the greateſt ſide is always oppoſite to the greateſt angle, and the leaſt ſide to the leaſt angle. The Demonſtrations of all theſe truths are exactly the ſame with thoſe in the correſponding caſes of plane triangles, ſo that it will be judged unneceſſary to ſpecify them in this place.

Theorem IV.

130. *If, from the three angles,* B, A *and* C, *of a ſpherical triangle,* BAC, *as poles, there be deſcribed upon the ſurface of the ſphere three arcs of great circles,* DF, DE *and* FE, *which, by their interſections, form another ſpherical triangle,* DEF; *each ſide of this new triangle will be the ſupplement to the angle which is at its pole, and each of its angles the ſupplement to that ſide in the triangle* BAC, *whereto it is oppoſite.* Fig. 11

DEMONSTRATION.

Let the ſides, AB, AC and BC, of the triangle BAC, be produced till they meet thoſe of the triangle DEF in the points, I, L; M, N; G, K: then, ſince the point A is the pole of the arc DILE, the diſtance of the points A and E will be 90°; and, ſince C is the pole of the arc EF, the points C and E will likewiſe be 90° diſtant; therefore, by *art.* 117, the point E is the pole of the arc AC. We might prove, in like manner, that F is the pole of BC, and D that of the arc AB.——This conſtruction

I being

being well underſtood, we ſhall have; $DL=90^\circ$, and $IE=90^\circ$; and therefore, $DL+IE$, or $DL+EL+IL$, or, $DE+IL=180^\circ$: conſequently, the arc DE is the ſupplement to the angle BAC, meaſured by the arc IL (*art.* 121). We might prove, in the ſame manner, that EF is the ſupplement to the angle BCA, meaſured by the arc MN, and that DF is the ſupplement to the angle ABC, meaſured by GK: whence it follows, that each ſide of the triangle DEF is the ſupplement to the angle in the triangle BAC, which is at its pole. Q. E. 1°. D.

2°. The angles of the ſame triangle are ſupplemental to the ſides of the triangle ABC: for, ſince the arcs AL and BG are each 90°, we ſhall have, $AL+BG$, or, $GL+AB=180^\circ$; but GL is the meaſure of the angle EDF, by *art.* 121, and conſequently AB the ſupplement thereof. We might prove in the ſame manner, that AC and BC are the ſupplements to the angles at E and F: wherefore all the angles of the triangle DEF are ſupplemental to the ſides oppoſite to them in the triangle BAC. Q. E. 2°. D.

SECT. II.

Of the Reſolution of right-angled ſpherical Triangles.

Preparation to the following Theorems.

Fig. 12. 131. Let GBPQ be a pyramid, compoſed of four right-angled triangles, GBQ, GBP, GPQ and BPQ; and let, AB, AC and BC, be three circular arcs, deſcribed from the centre G with the radius GB, and evidently forming a ſpherical triangle, BAC, right-angled at A; ſince the planes, GQP, GBP, are perpendicular to each other: then, if we make the radius equal to unity, we ſhall eaſily obtain the values ſpecified in the following Table,

Table, for all the parts of the said triangle; by recollecting that, *tang.* $= \frac{sin.}{cos.}$, and, *cot.* $= \frac{RR}{tang.} = \frac{cos.}{sin.}$

		sine		cos.		tang.
1°. The arc BC, or ang. QGB	hath for its *sine*	$\frac{BQ}{GQ}$	for its *cos.*	$\frac{BG}{GQ}$	for its *tang.*	$\frac{BQ}{BG}$*
2°. The arc BA, or ang. BGP		$\frac{BP}{GP}$		$\frac{GB}{GP}$		$\frac{BP}{BG}$
3°. The arc AC, or ang. PGQ		$\frac{QP}{GQ}$		$\frac{GP}{GQ}$		$\frac{QP}{GP}$
4°. The ang. ABC, or QBP		$\frac{PQ}{BQ}$		$\frac{BP}{BQ}$		$\frac{QP}{BP}$
5°. The ang. BCA		$\frac{BP \times GQ}{BQ \times GP}$		$\frac{QP \times BG}{GP \times BQ}$		$\frac{BP \times GQ}{QP \times BG}$

* The expressions in *cotangents*, being obtained by only inverting those of the *tangents*, are here omitted.

132. In order to ſhew how theſe ſeveral expreſſions are demonſtrated, it will be ſufficient to give the demonſtration of the firſt line only. And to do this, I aſſume any line, GR, which I regard as the ſine total of the Tables; then I let fall RS perpendicularly upon GB, and, it is evident that, this line will be the *ſine* of the arc BC, and GS the *coſine* thereof. This being premiſed, the ſimilar triangles, GRS, GQB, give the two following proportions; $QG : QB :: GR\ (1) : RS = \textit{ſin.}\,BC = \frac{QB}{QG}$, and, $QG : GB :: GR\ (1) : GS = \textit{coſ.}\,BC = \frac{GB}{QG}$; from whence I immediately deduce, $\textit{tang.}\ BC = \frac{BQ}{BG}$, and, $\textit{cot.}\ BC = \frac{BG}{BQ}$: the other expreſſions are demonſtrated exactly in the ſame manner.—— To prove the truth of the following Theorems, the particular values of the terms of the proportions to be demonſtrated need only be ſubſtituted, and there will be always found a perfect equality ſubſiſting between the product of the extremes and that of the means.

Theorem I.

Fig. 13. 133. *In any right-angled ſpherical triangle,* BAC; *the ſine total is to the ſine of the hypothenuſe, as the ſine of either angle is to that of its oppoſite leg; and contrarily.*

N. B. By this Theorem, the expreſſions of the *ſine* and *coſine* of the angle BCA were obtained; as well as thoſe of its *tangent* and *cotangent*.

Theorem II.

134. *In any right-angled ſpherical triangle we ſhall always have; as radius is to the coſine of either angle,*

ſo

so is the tangent of the hypothenuse to the tangent of the leg adjacent to this angle; that is, R : *cos.* B : : *tang.* BC : *tang.* AB, or, R : *cos.* C : : *tang.* BC : *tang.* AC.

Theorem III.

135. *We shall likewise have in every right-angled spherical triangle; as the sine total is to the cosine of one of the legs, so is the cosine of the other leg to that of the hypothenuse*; or, which is the same, R : *cos.* AB : : *cos.* AC : *cos.* BC.

Theorem IV.

136. *We shall moreover have; as radius is to the sine of either angle, so is the cosine of the adjacent leg to the cosine of the other angle*; or, R : *sin.* B or *sin.* C : : *cos.* AB or *cos.* AC : *cos.* C or *cos.* B.

Theorem V.

137. *As radius is to the sine of one of the legs, so is the tangent of its adjacent angle to the tangent of the other leg*; or, R : *sin.* AB : : *tang.* B : *tang.* AC, and, R : *sin.* AC : : *tang.* C : *tang.* AB.

Theorem VI.

138. *The radius is to the cotangent of one of the angles, as the cotangent of the other angle is to the cosine of the hypothenuse*; *or* (which comes to the same) *the radius is to the cosine of the hypothenuse, as the tangent of one angle is to the cotangent of the other angle*; that is, R : *cot.* B : : *cot.* C : *cos.* BC ; or, R : *cos.* BC : : *tang.* B : *cot.* C : : *tang.* C : *cot.* B.

Corollary I.

139. From the second Theorem it follows that, if two right-angled spherical triangles have one leg common;

common; the *tangents* of their hypothenuſes are in the inverſe ratio of the *coſines* of the angles adjacent to this leg.

COROLLARY II.

140. From the third Theorem it likewiſe follows that, if two right-angled ſpherical triangles have one leg common; the *coſines* of their hypothenuſes are as the *coſines* of their other legs.

COROLLARY III.

141. It alſo follows from the fourth Theorem that, if two right-angled ſpherical triangles have one leg common; the *coſines* of the angles oppoſite to this leg are to each other as the *ſines* of the adjacent angles.

COROLLARY IV.

142. It moreover follows from the fifth Theorem that, if two right-angled ſpherical triangles have one leg common; the *ſines* of their other legs are reciprocally as the *tangents* of the angles above theſe legs.

COROLLARY V.

143. Laſtly, if two right-angled ſpherical triangles have one angle common; the *ſines* of their hypothenuſes are as the *ſines* of their legs oppoſite to this angle, and the *tangents* of theſe legs as the *ſines* of the legs adjacent to the ſaid angle.—— The firſt of theſe truths is a direct conſequence of the firſt Theorem, and the ſecond of the fifth.†

SCHOLIUM.

144. The ſix preceding Theorems contain whatever is neceſſary for obtaining the ſolutions of

† See *Emerſon*'s Elements of Trigonometry, p. 158, *&c.* 2d edition.

all

all the poſſible caſes of right-angled ſpherical triangles; as we may be eaſily convinced by *Tab.* I. at the end of this Chapter. However, as there may be ſome difficulty in retaining, as well as ſome danger in confounding them, we ſhall alſo add the famous Theorem of *Neper*, whereby they are all reduced to two *general* caſes, that may be very eaſily remembered, provided we perfectly underſtand the following definitions; which are abſolutely neceſſary to convey a juſt idea of its nature.

DEFINITIONS.

145. When three parts of a right-angled ſpherical triangle are ſo ſituated, that two of them immediately touch the third, or are ſeparated from it by the right-angle only; theſe two parts are ſaid to be *adjacent* to the third, which is called the *mean* or *middle* part.

146. But when three of the parts of a right-angled ſpherical triangle are ſo ſituated, that between one of theſe parts, conſidered as the *mean*, and each of the other two, there is always found ſome other part of the ſame triangle; then are theſe two parts ſaid to be *ſeparated*.——The right-angle is not ſuppoſed to ſeparate its contiguous parts. —— This being premiſed;

If the *middle* parts be	the *adjacent* ones, or *extremes conjunct*, will be	and the *ſeparated* ones, or *extremes disjunct*,
AB	AC & B	BC & C.
AC	AB & C	BC & B.
BC	B & C	AC & AB.
B	AB&BC	AC & C.
C	AC&BC	AB & B.

GENERAL THEOREM.

147. *If, in any right-angled ſpherical triangle, the complements of the ſides containing the right-angle be ſubſtituted for the ſides themſelves, we ſhall always have; the*

the rectangle under the sine total and cosine of the middle *part, equal to the rectangle under the cotangents of the* adjacent *parts, or to that under the sines of the* separated *parts.*

DEMONSTRATION.

We have taken care to particularize in the Table for the solutions of right-angled triangles all the cases of this Theorem, denoting by A, the cases which have relation to the *adjacent* parts, and by S, those of the *separated* parts; so that the truth of the general Theorem is demonstrated by that of the particular Theorems corresponding to those cases. However, it may likewise be proved from the Table in *art.* 131, by substituting for each particular term the values therein specified: thus, for instance, if we would demonstrate the first line in *art.* 146, we need only prove that, $R \times$ *sin.* $AB =$ *tang.* $AC \times$ *cot.* $B =$ *sin.* $BC \times$ *sin.* C. Now, by taking the expressions of these lines, as given in *art.* 131, we shall get, $R \times \frac{BP}{GP} = \frac{QP \times BP}{GP \times QP} = \frac{BQ \times BP \times GQ}{GQ \times BQ \times GP}$; all which become manifestly the same, after expunging the terms which destroy in the two last fractions.——It would be the same with respect to the other lines; so that we may very justly conclude, that our Theorem is true in all possible cases. Q. E. D.

Of the values of the Angles of a right-angled spherical Triangle, with regard to the Sides which contain the right angle, &c.

THEOREM.

148. *In any right-angled spherical triangle,* BAC, or *b*AC, *the angles above the hypothenuse are always of*

of the same affection with their opposite sides; 2°, *the hypothenuse is less or greater than a quadrant, according as the legs containing the right-angle are of the same or different affection*; that is to say, *according as they are both acute or obtuse, or the one acute and the other obtuse.*

DEMONTSRATION.

If the sides AB and AC be each 90°; then, since the angle A is right by supposition, the points B and C will become the poles of the arcs AC and AB; and consequently, the angles B and C be both right, (as being measured by arcs which are supposed 90°); that is to say, of the same affection with their opposite sides—Moreover, if the legs AB and AC be both acute, the angles which are opposite to them will be so likewise; as may be easily proved thus. Let the side AC be produced to F, so that AF may be 90°; then, as the point F will, by *art.* 117, be the pole of the arc AB, the angle B will be also 90°; and consequently, the angle ABC, which is less than the angle ABF, necessarily acute. We might prove, in like manner, that the angle at C is acute, when the side AB, which is opposite to it, is acute.——It is equally manifest, that the angles B and C, in the triangle B*a*C, right-angled at *a*, are obtuse, when the sides, *a*B, *a*C, which are opposite to them, are obtuse. ——If one of the sides A*b* is obtuse, and the other side AC acute, as in the right-angled triangle *b*AC, the angle at C will be likewise obtuse, and that at *b* acute. For, having taken upon AB*b* the arc AG 90°, and drawn from the point G, to the point C, the arc GC, the angle ACG will be right, by *art.* 118. since G is the pole of the arc AC; from whence it necessarily follows, that the angle AC*b*

will be obtuse For the same reason, the angle abC, in the right-angled triangle baC, will be obtuse; since it is opposite to the obtuse side aC; and consequently, the supplement thereof AbC acute; that is, of the same affection with its opposite side. Therefore, in general, the angles above the hypothenuse are of the same affection with the sides which are opposite to them. Q. E. 1°. D.

2°. It is evident, that BC, whether considered as the hypothenuse of the triangle BAC, or that of the triangle BaC, is less than BF; and that the hypothenuse bC, in the triangle baC, is greater than bF: from whence it follows, that the hypothenuse of any right-angled spherical triangle is always less than 90°, when the two legs are of the same affection; and greater, when they are of different. Q. E. 2°. D.

It will easily appear, that the converse of this Theorem is true in all its parts, viz. that, if the angles above the hypothenuse are of the same or different affection, their opposite sides will be so likewise; and that the legs are of the same or different affection, according as the hypothenuse is less or greater than a quadrant; but one or both of them 90°, when it is exactly a quadrant.

Problem I.

Fig. 13. 149. *The hypothenuse*, BC, *of a right-angled spherical triangle*, BAC, *together with the sum or difference of the two legs*, AB *and* AC, *being given; to determine the triangle.*

SOLUTION.

Since we have, by *art.* 135, R : *cos.* AB : : *cos.* AC : *cos.* BC, we shall also have, *cos.* AB × *cos.* AC = R × *cos.* BC. But, by *art.* 104, *cos.* AB × *cos.* AC = $\frac{1}{2}$

$= \frac{1}{2}\, cof.\, \overline{AB+AC} + \frac{1}{2}\, cof.\, \overline{AB-AC}$; and, consequently, $2R \times cof.\, BC - cof.\, \overline{AB+AC} = cof.\, \overline{AB-AC}$, or, $2R \times cof.\, BC - cof.\, \overline{AB-AC} = cof.\, \overline{AB+AC}$. Q. E. I.

COROLLARY.

150. Hence, if the legs be equal, we ſhall have, $2\, cof.\, BC - R = cof.\, 2AB$* or $cof.\, 2AC$; which ſhews us, that in this caſe, the *coſine* of the double of either of the legs is equal to twice the *coſine* of the hypothenuſe leſs the ſine total.

PROBLEM II.

151. *Given one of the legs, and the ſum or difference of the hypothenuſe and other leg; to find the hypothenuſe.*

SOLUTION.

Since we have, by *art.* 135, $R : cof.\, AB :: cof.\, AC : cof.\, BC$, we ſhall likewiſe have, *componendo et dividendo*, $R + cof.\, AB : R - cof.\, AB :: cof.\, AC + cof.\, BC : cof.\, AC - cof.\, BC$; which, by *art.* 76 and 100, becomes, $cot.\, \frac{AB}{2} : tang.\, \frac{AB}{2} :: cot.\, \frac{BC+AC}{2} : tang.\, \frac{BC-AC}{2}$; from whence it follows, that, if we know the ſum or difference of the hypothenuſe and one leg, we ſhall be able by this analogy to find the difference or ſum of the hypothenuſe and other leg.

* Becauſe, $cof.\, AB \times cof.\, AC = R \times cof.\, BC$, then becomes, $cof.^2\, AB$ or $cof.^2\, AC = R \times cof.\, BC$: but, by *art.* 32, $cof.^2\, AB$ or $cof.^2\, AC = \frac{R^2 + R \times cof.\, 2AB}{2}$ or $\frac{R^2 + R \times cof.\, 2AC}{2}$; and therefore, $2\, cof.\, BC - R = cof.\, 2AB$ or $cof.\, 2AC$.

Problem III.

152. *One of the angles above the hypothenuse, with the sum or difference of the hypothenuse and adjacent leg in a right-angled spherical triangle, being given; to determine the triangle.*

SOLUTION.

Since we have, by *art.* 134, R : *cos.* B : : *tang.* BC : *tang.* AB, we shall also have, *componendo et dividendo*, $R + cos.\,B : R - cos.\,B :: tang.\,BC + tang.\,AB : tang.\,BC - tang.\,AB$; which, by *art.* 76 and 87, become, $cot.^2\,\tfrac{1}{2}B : R^2$, or, $cot.\,\tfrac{1}{2}B : tang.\,\tfrac{1}{2}B :: sin.\,\overline{BC+AB} : sin.\,\overline{BC-AB}$. Q. E. I.

Problem IV.

153. *Given the hypothenuse, and the sum or difference of the angles above the hypothenuse; to find these angles.*

SOLUTION.

By *art.* 138, we have, R : *cos.* BC : : *tang.* C : *cot.* B; which, by proceeding as in the last Problem, gives, $R^2 : tang.^2\,\tfrac{1}{2}BC :: cos.\,\overline{B-C} : cos.\,\overline{B+C}$;* from whence the solution required is easily obtained.——But, if the two angles be equal, we shall have, $R^2 : tang.\,\tfrac{1}{2}BC :: tang.\,\tfrac{1}{2}BC : cos.\,2B$ or $cos.\,2C$. Q. E. I.

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* Because, $R : cos.\,BC :: tang.\,C : cot.\,B :: tang.\,C : tang.\,\overline{90^\circ - B}$, we shall likewise have, $R + cos.\,BC : R - cos.\,BC :: tang.\,C + tang.\,\overline{90^\circ - B} : tang.\,C - tang.\,\overline{90^\circ - B}$; and consequently, by *art.* 76 and 87, $R^2 : tang.^2\,\tfrac{1}{2}BC :: sin.\,\overline{C + 90^\circ - B} : sin.\,\overline{C - 90^\circ + B} :: cos.\,\overline{B-C} : cos.\,\overline{B+C}$.

Pro-

PROBLEM V.

154. *Given in two right-angled ſpherical triangles,* BAC, BDG, *which have one angle common, the legs,* AC *and* DG, *oppoſite to this angle, with the ſum or difference of their hypothenuſes; to determine theſe triangles.* Fig. 14.

SOLUTION.

Since, by *art.* 143, *ſin.* DG : *ſin.* AC : : *ſin.* BG : *ſin.* BC, we ſhall likewiſe have, *componendo et dividendo*, *ſin.* DG + *ſin.* AC : *ſin.* DG — *ſin.* AC : : *ſin.* BG + *ſin.* BC : *ſin.* BG — *ſin.* BC, and conſequently, by *art.* 95, $tang. \frac{DG+AC}{2} : tang. \frac{DG-AC}{2} : : tang. \frac{BG+BC}{2} : tang. \frac{BG-BC}{2}$; from whence the ſolution of the Problem will be very eaſy, ſince three terms in this proportion will always be known. Q. E. I.

PROBLEM VI.

155. *Let there be ſtill two right-angled ſpherical triangles, which have one angle common, and let the legs oppoſite to this angle with the ſum or difference of the adjacent legs be given; to determine the triangles.*

SOLUTION.

Since we have, by *art.* 143, *tang.* DG : *tang.* AC : : *ſin.* BD : *ſin.* BA, we ſhall likewiſe have, *tang.* DG + *tang.* AC : *tang.* DG — *tang.* AC : : *ſin.* BD + *ſin.* BA : *ſin.* BD — *ſin.* BA; from whence we deduce, by making ſubſtitutions ſimilar to the preceding, $ſin. \overline{DG+AC} : ſin. \overline{DG-AC} : : tang. \frac{BD+BA}{2} : tang. \frac{BD-BA}{2}$ †. Q. E. I.

† See *Simpſon*'s Trigonometry, p. 69, &c.

SECT.

SECT. III.

Containing the Resolution of oblique-angled spherical Triangles.

THEOREM I.

Fig. 15. 156. *In any spherical triangle,* BAC, *the sines of the angles are to each other directly as the sines of their opposite sides.*

DEMONSTRATION.

From any angle, A, of the spherical triangle BAC, let fall the arc AD perpendicularly upon the base BC: then, in the right-angled spherical triangles, BAD, CAD, we shall have, by *art.* 133, R : *sin.* AB : : *sin.* B : *sin.* AD, and, R : *sin.* AC : : *sin.* C : *sin.* AD; whence we immediately deduce, *sin.* AB × *sin.* B = *sin.* AC × *sin.* C; and consequently this proportion; *Sin.* B : *sin.* C : : *sin.* AC : *sin.* AB. Q. E. D.

DEFINITIONS.

157. The angles, BAD, CAD, which the sides, AB, AC, containing the angle BAC, form with the perpendicular, are called the *segments of the vertical angle*; whether the perpendicular falls within or without the triangle BAC.

2°. In like manner, the parts, BD, DC, of the side BC, contained between the points, B, C, and the point D, where the perpendicular AD meets this side, are called the *segments of the base*; whether the base be produced or not.

3°. With regard to the segments, BAD, CAD, of the angle BAC, the containing sides, AB, AC, will be called *adjacent parts*, as being so in effect; but the angles, B, C, above the base BC, *separated parts*,

parts, ſince between theſe angles and the ſaid ſegments are found the ſides, BA, CA.

4°. With reſpect to the ſegments, BD, CD, of the baſe, the angles, B, C, will be *adjacent parts*, and the ſides, BA, CA, *ſeparated parts.*—— This being premiſed, it will be eaſy to remember the two parts of the following Theorem.

Theorem II.

158. *If from any angle*, A, *of a ſpherical triangle*, BAC, *we let fall a perpendicular*, AD, *upon the oppoſite ſide*, BC, *(produced if neceſſary)*; *we ſhall always have*;

1°. *The ſines of the ſegments of this angle as the coſines of the ſeparated parts*; *and the coſines thereof as the cotangents of the adjacent parts.*

2°. *The ſines of the ſegments of the baſe as the cotangents of the adjacent parts*; *and the coſines thereof as the coſines of the ſeparated parts*:

that is
1°. *Sin.* BAD : *ſin.* CAD : : *coſ.* B : *coſ.* C.
2°. *Coſ.* BAD : *coſ.* CAD : : *cot.* AB : *cot.* AC.
3°. *Sin.* BD : *ſin.* CD : : *cot.* B : *cot.* C.
4°. *Coſ.* BD : *coſ.* CD : : *coſ.* AB : *coſ.* AC.

DEMONSTRATION.

In the right-angled ſpherical triangle BDA we ſhall have, by *Theorem* IV. R : *coſ.* AD : : *ſin.* BAD : *coſ.* B; and likewiſe, in the right-angled triangle CDA, by the ſame, R : *coſ.* AD : : *ſin.* CAD : *coſ.* C; therefore, *Sin* BAD : *ſin.* CAD : : *coſ.* B : *coſ.* C. Q. E. 1°. D.

2°. The ſame triangles will alſo give, by *Theorem* II. R : *coſ.* BAD : : *cot.* AD : *cot.* AB, and, R : *coſ.* CAD : : *cot.* AD : *cot.* AC; therefore, ſince theſe two proportions have the ſame antecedents, we

we shall have, *Cos.* BAD : *cos* CAD : : *cot.* AB : *cot.* AC. Q. E. 2°. D.

3°. The same triangles will moreover give, by *Theorem* V. the two following proportions ; R : *sin.* BD : : *cot.* AD : *cot.* B, and, R : *sin.* CD : : *cot.* AD : *cot.* C ; therefore, since the antecedents are equal, the consequents will be likewise proportional, and give, *Sin.* BD : *sin.* CD : : *cot.* B : *cot.* C. Q. E. 3°. D.

Lastly, by *Theorem* III. we shall have in the triangle BAD, R : *cos.* BD : : *cos.* AD : *cos.* AB ; and in the triangle CDA, R : *cos.* DC : : *cos.* AD : *cos* AC ; from whence it follows, that *Cos.* BD : *cos.* CD : : *cos.* AB : *cos.* AC. Q. E. 4°. D.

SCHOLIUM.

159. The four parts of this Theorem are, upon the whole, of equal import with the Corollaries, which we have subjoined to the six Theorems upon right-angled triangles ; only, we have here attempted to render them somewhat easier to be remembered, by means of the definitions which we have prefixed to this Theorem.

Preparation to the following Theorems.

PROBLEM.

Fig. 16. 160. *Any spherical triangle,* BAC, *being given, one side whereof,* AB, *is supposed to be upon the circumference of a great circle,* ABRF*ar* ; *it is required to find the orthographic projection of this triangle upon the plane of the said circle* ; that is to say, *that which is formed by lines, let fall from all the points of the sides of the triangle,* ABC, *perpendicularly upon the plane,* ABRF*ar.*

SOLU-

SOLUTION.

From the extremities, A, B, of the arc AB, let there be drawn to the centre G the radii, GA, GB; alſo, through this centre (which is likewiſe that of the ſphere), let a plane or great circle, *r*DR*o*, be conceived to paſs perpendicularly to the plane AR*ar*, ſo that their common ſection, R*r*, may be perpendicular to the radius AG; laſtly, let the arc AC be produced till it meet the circumference *r*DR*o*, in the point D; from whence, let the line DG be drawn to the centre G, as alſo the line D*d* perpendicular to the diameter R*r*. This being done, it is evident, that the angle DGR is equal to the angle BAC, which is formed by the interſection of the planes, BAG, CAG, (ſince the lines, DG, RG, are both perpendicular to their common ſection, AG); and, that the angle DG*r* is equal to the ſupplement thereof. ——— It is equally manifeſt, that, if from the point C we let fall a line, C*c*, perpendicularly upon the plane AR*ar*, the point *c* will be the projection of the angular point C. Moreover, if through the point C the plane of a little circle, *l*CL, be made to paſs in a poſition parallel to the plane *r*DR*o*, the common ſection, *l*L, of this plane, and that of the great circle AR*ar*, will be alſo perpendicular to the radius AG, and determine each of the arcs, AL, A*l*, equal to the arc AC; and the projection of the point C will be found in a point of this line: but it will be likewiſe found (for the ſame reaſon) in a line, F*f*, which is the common ſection of the plane of the great circle AR*ar*, and that of the little circle *f*CF perpendicular thereto; provided this line be alſo perpendicular to the radius BG, and of courſe determine each of the arcs, BF, B*f*,

equal to the arc BC: and therefore, when the three sides of a triangle, BAC, are known, it is easy to perceive, from hence, how the projection of an angle C may be determined by the intersection of the lines, L*l*, F*f*, upon the plane of the circle ABR*ar*. Q. E. I. *et* D.

COROLLARY I.

161. The right-angled triangles, D*d*G, C*c*H, being similar, on account of the parallel lines whereof they are composed, we shall have, DG : CH or *r*G : *l*H : : *d*G : *c*H; that is to say, R : *sin.* AC : : *cos.* BAC : *c*H; therefore, as we should have the same proportion for all the projected points of the arc AC, it follows, that the projection of this arc upon the plane of the circle ABR*ar* is an ellipse;* whereof AG is half the transverse axis, and *d*G half the conjugate.

COROLLARY II.

162. It likewise follows, from the preceding Corollary, that the *orthographic* projection of a circle, or part of a circle, is always an ellipse, or part of an ellipse; whereof half the transverse axis is equal to the *sine total*, and half the conjugate to the *cosine* of the angle contained between the plane of the said circle and the plane of projection.

SCHOLIUM.

163. In the two following Chapters, we shall give a more minute Theory of the *orthographic* pro-

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* See *Robertson*'s Translation of *De la Caille*'s Astronomy, p. 160, &c.

jection,

jection, and of its application both to the graphical and analytical resolution of all the cases of spherical triangles. What we have said thereon, in this place, must be considered only as a *Lemma* absolutely necessary for the perfect understanding of the following Theorem; the difficulty whereof will be very easily surmounted, provided we have conceived a just idea of the nature of the planes, mentioned in the foregoing construction.

THEOREM III.

164. *In any spherical triangle,* BAC, *this proportion will always be true: as the product of the sines of the sides,* AB, AC, *containing any angle,* BAC, *is to the product of the sines of the differences of these and half the sum of the three sides, so is radius squared to the square of the sine of half the said angle;* or, which is the same, *Sin.* AB×*sin.* AC : *sin.* $\frac{AB+AC+BC}{2}$—AC×*sin.* $\frac{AB+AC+BC}{2}$—AB : : R^2 : *sin.*^{2} $\frac{1}{2}$ BAC. Fig. 17.

Construction necessary to the Demonstration.

Upon the plane of the circle ABR*ar*, and on each side of the point A, let there be taken the arcs, AL, A*l*, equal to the arc AC; likewise, on each side of the point B, let the arcs, BF, B*f*, be taken equal to the arc BC, and afterwards the chords, L*l*, F*f*, drawn respectively perpendicular to the radii, GA, GB: then (from what hath been demonstrated in the preceding Problem) it is evident, that the intersection C of these chords will be the projection of the angle C, in the triangle BAC; and (from this construction) that, BL=AC—AB; B*l*=AC+AB; LF=BF or BC—AC+AB; *lf*=AB+AC—BC, and, L*f*=BC+AC—AB. In the next place, let this Proportion be made, H*l* : CH : :

$Gr : Gd$; or, which is the same, upon Gr, let there be taken, Gd, a fourth proportional to the three lines, Hl, CH, Gr, and this line will be the *cosine* of the angle BAC, as appears from *fig.* 16; and consequently, if at the point d we raise a right line Dd, perpendicular to the radius Gr, it will determine the angle RGD equal to the angle BAC, and be the *sine* thereof. Lastly, from the point D, to the extremities of the diameter Rr, let there be drawn the chords, DR, Dr; upon which from the centre G let fall the perpendiculars, GS, Gs, and then from the points, S, s, the perpendiculars, SV, su, upon the diameter Rr; and it will manifestly appear, from this construction, that RS is the *sine* of half the angle BAC, and rs the *sine* of half its supplement; that is, the *cosine* of half the said angle.——So much being premised, we shall find no difficulty in the

DEMONSTRATION.

In the right-lined triangle CLF, the *sines* of the angles being to each other as the halves of their opposite sides, and the angle, LCF*, being manifestly equal to the angle AGB, its *sine* will be equal to that of the arc AB, and we shall have this proportion; $\sin. C$ or $\sin. AB : \sin. F :: \frac{1}{2} LF : \frac{1}{2} CL$; also, on account of the proportional lines, GR, HL, Gd, CH, we shall have, $HL : GR :: \frac{1}{2} CL : VR = \frac{1}{2} Rd$; and, likewise, since the lines, GR, RS, and RV, are in continued proportion, $GR : VR :: \overline{GR}^2 : \overline{RS}^2$: then, if we multiply the corresponding terms of these three proportions

* Because it is measured by half the sum of the arcs, LF, lf; which, by what hath been said above, are equal to 2 AB.

together,

together, and expunge such quantities as are common both to the antecedents and consequents, we shall get, $sin.\ AB \times HL : sin.\ F :: \frac{1}{2}$ $LF \times \overline{GR}^2 : \overline{RS}^2$, or, $sin.\ AB \times HL : sin.\ F \times \frac{1}{2} LF :: \overline{GR}^2 : \overline{RS}^2$; but, $HL = sin.\ AC$; $sin. F = sin.\ \frac{1}{2} LAf$ $= sin.\ \frac{BC+AC-AB}{2} = sin.\ \frac{\overline{BC+AC+AB}}{2} - AB$, and, $\frac{1}{2}$ $LF = sin.\ \frac{1}{2} LRF = sin.\ \frac{BC+AB-AC}{2} = sin.$ $\frac{\overline{BC+AB+AC}}{2} - AC$; and therefore, if we substitute these values in the last proportion, it will become,

$$Sin.\ AB \times sin.\ AC : sin.\ \frac{\overline{BC+AB+AC}}{2} - AB \times sin.\ \frac{\overline{BC+AB+AC}}{2} - AC :: R^2 : sin.^2\ \tfrac{1}{2} BAC.$$ Q. E. D.

Theorem IV.

165. *Let the same construction be supposed as in the preceding Theorem, and I say that we shall likewise have this proportion; as the product of the sines of the sides,* AB, AC, *containing any angle,* BAC, *is to the product of the sine of half the difference of these sides and the third by the sine of half the sum of the three sides, so is radius squared to the square of the cosine of half the included angle*; that is, $Sin.\ AB \times sin.\ AC : sin.\ \frac{AB+AC-BC}{2} \times sin.\ \frac{AB+AC+BC}{2} :: R^2 : cos.^2$ $\frac{1}{2}$ BAC.

DEMONSTRATION.

In the triangle Clf, the *sines* of the angles are to each other as the halves of their opposite sides: but, it is evident that, the angle at f is measured by half the arc FDl, which is the supplement to half

half the arc FAl, equal to half the sum of the three sides, AC, AB, BC: the arc lf is also manifestly equal to, AB+AC—BC; and therefore, the half of its chord will be the *sine* of, $\frac{AB+AC-BC}{2}$. This being premised, we shall have, *sin.* AB : *sin.* f : : $\frac{1}{2} lf$: $\frac{1}{2}$ Cl; by construction, Hl : Gr : : $\frac{1}{2}$ Cl : $ur = \frac{1}{2} dr$; and likewise, since the lines, Gr, rs, and ru, are in continued proportion, Gr : ur : : $\overline{Gr}^2 : \overline{rs}^2$; therefore, if we multiply the corresponding terms together, we shall get, *sin.* AB×Hl : *sin.* f : : $\frac{1}{2} lf \times \overline{Gr}^2 : \overline{rs}^2$, or, *sin.* AB × H$l$: : $\frac{1}{2} lf \times$ *sin.* f : : $\overline{Gr}^2 : \overline{rs}^2$; and, by substituting the value of each line, *Sin.* AB×*sin.* AC : *sin.* $\frac{AB+AC-BC}{2} \times$ *sin.* $\frac{AB+AC+BC}{2}$: : R^2 : *cos.*^{2} $\frac{1}{2}$ BAC. Q. E. D.

COROLLARY I.

166. Hence, if we call the sum of the three sides, s; the side opposite to the angle sought, a; and the sides including this angle, b, and c; we shall have; *sin.* $\frac{1}{2}$ angle $= \frac{r\sqrt{sin.\,\frac{1}{2}s-b \times sin.\,\frac{1}{2}s-c}}{\sqrt{sin.\,b \times sin.\,c}}$, and,

$$cos.\ \tfrac{1}{2}\ angle = \frac{r\sqrt{sin.\,\frac{b+c-a}{2} \times sin.\,\frac{b+c+a}{2}}}{\sqrt{sin.\,b \times sin.\,c}} = \frac{r\sqrt{sin.\,\frac{1}{2}s-a \times sin.\,\frac{1}{2}s}}{\sqrt{sin.\,b \times sin.\,c}}.$$

COROLLARY II.

167. Since we have, by *art.* 69, $\frac{sin.\,A \times r}{cos.\,A}$ = *tang.* A, it follows, that we shall also have, *tang.* $\frac{1}{2}$ angle

gle $= \frac{r\sqrt{fin.\,\overline{\frac{1}{2}s-b}\times fin.\,\overline{\frac{1}{2}s-c}}}{\sqrt{fin.\,\overline{\frac{1}{2}s-a}\times fin.\,\frac{1}{2}s}}$, and, *cot.* $\frac{1}{2}$ angle $=$
$\frac{r\sqrt{fin.\,\overline{\frac{1}{2}s-a}\times fin.\,\frac{1}{2}s}}{\sqrt{fin.\,\overline{\frac{1}{2}s-b}\times fin.\,\overline{\frac{1}{2}s-c}}}$.

Corollary III.

168. Since we have, by *art.* 71, *fin.* 2 A $=$ $\frac{cof.\,A\times 2\,fin.\,A}{R}$, we ſhall alſo have the *ſine* of the whole angle A $= \frac{2r\sqrt{fin.\,\frac{1}{2}s\times fin.\,\overline{\frac{1}{2}s-a}\times fin.\,\overline{\frac{1}{2}s-b}\times fin.\,\overline{\frac{1}{2}s-c}}}{fin.\,b\times fin.\,c}$: which formula may be otherwiſe found by a comparison of the ſimilar triangles, *l*C*f*, LCF, &c. ——The formula, *coſ.* 2A $= \frac{2\,cof.^2\,A-R^2}{R}$, will give the *coſine*, likewiſe, of the whole angle A $=$ $\frac{\overline{2\,fin.\,\frac{1}{2}s\times fin.\,\overline{\frac{1}{2}s-a}-fin.\,b\times fin.\,c}\times r}{fin.\,b\times fin.\,c}$; and, if we divide the former of theſe two expreſſions by the latter, we ſhall obtain an expreſſion, (though ſomewhat more complicated), for the *tangent* of the whole angle.

Corollary IV.

169. If the ſides, including the angle ſought, be equal, we ſhall have; *ſin.* $\frac{1}{2}$ angle $= \frac{R\times fin.\,\frac{1}{2}BC}{fin.\,AB}$, and, *coſ.* $\frac{1}{2}$ angle $= \frac{r\sqrt{fin.\,\overline{b+\frac{1}{2}a}\times fin.\,\overline{b-\frac{1}{2}a}}}{fin.\,b}$.

SCHOLIUM.

170. From the ſeveral formulæ, which we have deduced from the two laſt Theorems, it will readily appear, that the reaſon of our chuſing to find one

one of the angles of a triangle, whereof we know the three ſides, by the *ſine* or *coſine* of its half, is, becauſe theſe formulæ are the moſt eaſily conſtructed by the *logarithms*; however, it muſt be acknowledged, that the formulæ, which we have given for the *tangent* and *cotangent* of half the angle ſought, may be likewiſe very advantageouſly uſed, as being very little inferior in ſimplicity to the other two; and therefore, we have *four* eaſy and compendious methods of ſolving this particular caſe of Trigonometry.

Neper, *Oughtred*, *Jones*, and ſome other *Engliſh* Mathematicians, in order to preſerve a greater analogy between the ſolutions of this caſe by the *ſine* and *coſine* of half an angle, have given formulæ ſomewhat different from ours with regard to the expreſſions; which, as they ſeem pretty well adapted to remembrance, we ſhall here ſubjoin. Let *a* and *c* be put for the ſides, containing the angle ſought, the ſum whereof call, *s*, and difference, *d*, and let *b* repreſent the ſide oppoſite to this angle, (which we ſhall regard as the baſe), and we ſhall have;

$$\textit{Sin.}\ \tfrac{1}{2}\ \text{angle} = \frac{r\sqrt{\textit{ſin.}\frac{b+d}{2}\times\textit{ſin.}\frac{b-d}{2}}}{\sqrt{\textit{ſin.}\ a\times\textit{ſin.}\ c}}\ ;\ \textit{coſ.}\ \tfrac{1}{2}\ \text{angle} = \frac{r\sqrt{\textit{ſin.}\frac{s+b}{2}\times\textit{ſin.}\frac{s-b}{2}}}{\sqrt{\textit{ſin.}\ a\times\textit{ſin.}\ c}}\ ;$$

and conſequently, *tang.* $\frac{1}{2}$

$$\text{angle} = \frac{r\sqrt{\textit{ſin.}\frac{b+d}{2}\times\textit{ſin.}\frac{b-d}{2}}}{\sqrt{\textit{ſin.}\frac{s+b}{2}\times\textit{ſin.}\frac{s-b}{2}}}.$$

If we would apply the preceding formulæ to plane trigonometry, it need only be conſidered that

that the *sines* of the sides of a triangle then become the sides themselves; and we shall have, by still preserving the same denominations for the three sides of a triangle; *sin.* $\frac{1}{2}$ angle $=$

$$\frac{r\sqrt{\overline{\tfrac{1}{2}s-b}\times\overline{\tfrac{1}{2}s-c}}}{\sqrt{bc}};\ cos.\ \tfrac{1}{2}\ \text{angle}=\frac{r\sqrt{\tfrac{1}{2}s\times\overline{\tfrac{1}{2}s-a}}}{\sqrt{bc}},\ \text{and,}$$

$$tang.\ \tfrac{1}{2}\ \text{angle}=\frac{r\sqrt{\overline{\tfrac{1}{2}s-b}\times\overline{\tfrac{1}{2}s-c}}}{\sqrt{\tfrac{1}{2}s\times\overline{\tfrac{1}{2}s-a}}}$$

: from whence we may perceive, that if, to the logarithms of the factors of the numerator of each fraction, we add the arithmetical complements of the factors of the denominator, and take half the sum thence arising, we shall obtain the angle sought, much more simply than by the common rules of plane Trigonometry.

It may be moreover observed, that the formula found for the *sine* of A, in *art.* 168, furnishes us with a pretty remarkable property of a plane triangle; viz. that the *sine* of any of its angles is equal to twice the area of this triangle, divided by the product of the sides which include the said angle: for it is demonstrated, in all the Elements of Geometry, that the area of a triangle is $=$ $\sqrt{\tfrac{1}{2}s\times\overline{\tfrac{1}{2}s-a}\times\overline{\tfrac{1}{2}s-b}\times\overline{\tfrac{1}{2}s-c}}$; which becomes the numerator of the formula for *sin.* A, when the triangle is right-lined.

THEOREM V.

171. *In any spherical triangle,* BAC, *whereof the three angles are known, we shall always have these two analogies for finding one of its sides;* 1°, *as the product of the sines of the angles above the side sought is to the product of the cosines of the differences of these angles and half the sum of the three angles, so is radius squared to the*

the square of the cosine of half the side sought; that is, *Sin.* $B \times$ *sin.* $C : cos.\overline{\frac{A+B+C}{2} - B} \times cos.\overline{\frac{A+B+C}{2} - C} :: R^2 : cos.^2 \frac{1}{2} BC$; and, 2°, *as the product of the sines of the angles adjacent to the side sought is to the product of the cosine of half the difference between these two angles and the third and half the sum of the three angles, so is radius squared to the square of the sine of half the side sought*; that is to say, *Sin.* $B \times$ *sin.* $C : cos. \frac{B+C-A}{2} \times cos. \frac{B+C+A}{2} :: R^2 : sin.^2 \frac{1}{2} BC$.

DEMONSTRATION.

Fig. 11. In the triangle DEF, all the parts whereof are the supplements to those in the triangle BAC, by *art.* 130, we shall have, by *Theorem* III. *sin.* $FD \times$ *sin.* $FE : sin\overline{\frac{DF+FE+DE}{2} - DF} \times sin.\overline{\frac{DF+FE+DE}{2} - FE} :: R^2 : sin.^2 \frac{1}{2} DFE$. But, the arcs, FD, FE, are the supplements to the angles, B, C, in the triangle BAC; and consequently, their *sines* the same as those of the said angles. Moreover, since the *sine* of half the supplement of any arc or angle is equal to the *cosine* of half such arc or angle; the second term will, after making proper substitutions, become, $cos.\overline{\frac{B+C+A}{2} - B} \times cos.\overline{\frac{B+C+A}{2} - C}$. In like manner, the angle DFE being the supplement to the side BC, its half will be the complement of half this side, and the *sine* of such half the *cosine* of half the said side; and therefore, the preceding proportion will now be changed into this; *Sin.* $B \times$ *sin.* $C : cos.\overline{\frac{B+C+A}{2} - B} \times cos.\overline{\frac{B+C+A}{2} - C} :: R^2 : cos.^2 \frac{1}{2} BC$. Q. E. 1°. D.

2°. In

2°. In the same triangle, DEF, we shall have, by *Theorem* IV. $\sin.\ FD \times \sin.\ EF : \sin.\ \frac{DF+EF-DE}{2} \times \sin.\ \frac{DF+EF+DE}{2} :: R^2 : \cos.^2 \frac{1}{2} DFE$; which proportion, by making substitutions similar to those which we made in the preceding case, will be changed into, $Sin.\ B \times \sin.\ C : \cos.\ \frac{B+C-A}{2} \times \cos.\ \frac{B+C+A}{2} :: R^2 : \sin.^2 \frac{1}{2} BC$. Q. E. 2°. D.

Corollary I.

172. Hence, if we call the sum of the three angles of any triangle, s; these three angles, α, β, γ, respectively, and suppose, β, γ, to be the angles adjacent to the side sought, we shall have the following formulae; $\sin.\ \frac{1}{2}$ side sought $= \frac{r\sqrt{\cos.\ \frac{1}{2}s \times \cos.\ \overline{\frac{1}{2}s-\alpha}}}{\sqrt{\sin.\ \beta \times \sin.\ \gamma}}$; $\cos.\ \frac{1}{2}$ side $= \frac{r\sqrt{\cos.\ \overline{\frac{1}{2}s-\beta} \times \cos.\ \overline{\frac{1}{2}s-\gamma}}}{\sqrt{\sin.\ \beta \times \sin.\ \gamma}}$, and, $tang.\ \frac{1}{2}$ side $= \frac{r\sqrt{\cos.\ \frac{1}{2}s \times \cos.\ \overline{\frac{1}{2}s-\alpha}}}{\sqrt{\cos.\ \overline{\frac{1}{2}s-\beta} \times \cos.\ \overline{\frac{1}{2}s-\gamma}}}$.——— It would be easy to find formulae for the *sine*, *cosine*, *tangent*, and *cotangent*, of the whole side sought; but, as these expressions would be much more complicated than those above given, it will be unnecessary to specify them.

Corollary II.

173. When the two angles above the side sought are equal, we shall have by the first part of the Theorem; $\sin.^2 B : \cos.^2 \frac{1}{2} A :: R^2 : \cos.^2 \frac{1}{2} BC$,

 and,

and, by extracting the square roots, $cos. \frac{1}{2} BC = \frac{R \times cos. \frac{1}{2} A}{sin. B}$.——— The formula, which would arise from the second part, would not be quite so simple.

SCHOLIUM.

174. These Theorems contain the solutions of all the possible cases of oblique-angled spherical triangles, as we may be easily convinced from an inspection of the second *Table*, where all these solutions are collected and united. However, that we may give to this part all the extent and generality whereof it is susceptible, we shall yet annex another Section; in which we shall give several other very important and general Theorems, and in particular the famous analogies of *Neper*; wherein this Geometer's design seems to have been, to reduce the analogies of spherical Trigonometry to those of plane. But, by rendering to these Theorems all the generality requisite, it will be easy to observe, that the analogies of plane, are but particular cases of those of spherical, Trigonometry; and therefore, sufficient (in our opinion) to deduce them separately from thence by simple inferences only. It is surprising that most writers upon Trigonometry have made no mention of these analogies; whilst others have given them without demonstration, and others deformed them, (but Mr. *Wolf* in particular), by substituting in their room proportions considerably different from those of *Neper*.——Besides, these analogies may prove to be of very peculiar service in this; that they will greatly contribute to an easy retention of all the possible cases of oblique-angled triangles, whether spherical or right-lined.

SECT.

SECT. IV.

Containing a Demonstration of the analogies of Neper, *as well as of some others, both in plane and spherical Triangles not commonly given.*

THEOREM I.

175. *If from any angle*, A, *of an oblique-angled spherical triangle*, BAC, *we let fall a perpendicular*, AD, *upon the opposite side or base*, BC, *(produced if necessary), we shall always have this analogy; as the tangent of half the base is to the tangent of half the sum of the other two sides, so is the tangent of half the difference of these sides to the tangent of half the difference, or half the sum, of the segments of the base formed by the perpendicular, according as it falls within or without the triangle*; that is to say, *Tang.* $\frac{1}{2}$BC : *tang.* $\frac{AB+AC}{2}$ :: *tang.* $\frac{AB-AC}{2}$: *tang.* $\frac{BD\mp DC}{2}$. Fig. 15.

DEMONSTRATION.

We have proved in *art.* 158 that, *cos.* BD : *cos.* CD :: *cos.* AB : *cos.* AC, and therefore, we shall also have, *componendo et dividendo*, *cos.* BD+*cos.* CD : *cos.* BD—*cos.* CD :: *cos.* AB+*cos.* AC : *cos.* AB—*cos.* AC; but, by *art.* 100, *cos.* BD+*cos.* CD : *cos.* BD—*cos.* CD :: *cot.* $\frac{BD+CD}{2}$ *tang.* $\frac{BD-CD}{2}$; and for the same reason, *cos.* AB+*cos.* AC : *cos.* AB—*cos.* AC :: *cot.* $\frac{AB+AC}{2}$: *tang.* $\frac{AB-AC}{2}$: hence, since the two first terms of these two proportions are proportional, the two last will be so likewise, and give, *cot.* $\frac{BD+CD}{2}$: *tang.* $\frac{BD-CD}{2}$ :: *cot.* $\frac{AB+AC}{2}$: *tang.* $\frac{AB-AC}{2}$; which, after substituting for the *cotangents*

gents their values in the *tangents*, and multiplying extremes and means, produces, $\frac{tang.\,\overline{\frac{1}{2}AB-\frac{1}{2}AC}}{tang.\,\overline{\frac{1}{2}BD+\frac{1}{2}CD}} = \frac{tang.\,\overline{\frac{1}{2}BD-\frac{1}{2}CD}}{tang.\,\overline{\frac{1}{2}AB+\frac{1}{2}AC}}$. Now, if the perpendicular falls within the triangle, $\frac{BD+CD}{2}$ will be $=\frac{BC}{2}$, but if without, $\frac{BD-CD}{2}=\frac{BC}{2}$; and therefore, the laſt equation reduced into proportion will, to correſpond to both caſes, ſtand thus; *Tang.* $\frac{1}{2}$ BC : *tang.* $\frac{AB+AC}{2}$: : *tang.* $\frac{AB-AC}{2}$: *tang.* $\frac{BD\mp CD}{2}$. Q. E. D.

SCHOLIUM.

176. If we ſuppoſe the triangle right-lined, it is eaſy to perceive that, as the *tangents* then become the ſides themſelves, we ſhall have this proportion; *As the baſe of the triangle is to the ſum of the other two ſides, ſo is their difference to the difference or ſum of the ſegments formed by the perpendicular let fall upon the baſe, according as it falls within or without the triangle.*———It is likewiſe manifeſt, that this formula may be applied both to ſpherical and plane trigonometry, in order to reduce a triangle whoſe three ſides are given into two other right-angled ones; in each of which we ſhall know, beſides the right angle, the hypothenuſe and one leg. Yet (whether the triangle be ſpherical or right-lined) the formulæ in *art.* 164 and 165 ought (in our opinion) to be always preferred, as conducing to the eaſieſt and ſhorteſt ſolution.

THEOREM II.

Fig. 15. 177. *If we ſtill ſuppoſe that from any angle,* A, *of a ſpherical triangle,* BAC, *there be let fall a perpendicular,*

dicular, AD, *upon the opposite side, we shall have the two following proportions*; 1°, *as the sine of the sum of the angles above the side on which the perpendicular falls is to that of their difference, so is the tangent or cotangent of half the said side to the tangent of half the difference, or cotangent of half the sum, of its segments formed by the perpendicular*; that is, *Sin.* $\overline{C+B}$: *sin.* $\overline{C-B}$: : *tang.* or *cot.* $\frac{1}{2}$ BC : *tang.* or *cot.* $\frac{BD\mp CD}{2}$; and 2°, *as the sine of the sum of the other two sides is to that of their difference, so is the cotangent of half their included angle to the tangent of half the difference, or half the sum, of the segments of this angle*; that is to say, *Sin.* $\overline{AB+AC}$: *sin.* $\overline{AB-AC}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{BAD\mp CAD}{2}$.

DEMONSTRATION.

In the triangle BAC we have, by *art.* 158, *tang.* C : *tang.* B : : *sin.* BD : *sin.* CD, and therefore, we shall also have, *componendo et dividendo*, *tang.* C + *tang.* B : *tang.* C — *tang.* B : : *sin.* BD + *sin.* CD : *sin.* BD — *sin.* CD; but the relation of the two first terms is, by *art.* 87, equal to that of, *sin.* $\overline{C+B}$: *sin.* $\overline{C-B}$, and the relation of the two second, by *art.* 95, equal to that of, *tang.* $\frac{BD+CD}{2}$: *tang.* $\frac{BD-CD}{2}$; therefore, if these relations be substituted for the preceding, and *tang.* $\frac{1}{2}$ BC be put for *tang.* $\frac{BD+CD}{2}$, when the perpendicular falls within the triangle; but for *tang.* $\frac{BD-CD}{2}$, when it falls without, we shall have for the two cases; *Sin.* $\overline{C+B}$:

$\overline{C+B} : \mathit{sin.}\ \overline{C-B} :: \mathit{tang.}\ \frac{1}{2} BC : \mathit{tang.}\ \frac{BD-CD}{2} ::$ $\mathit{tang.}\ \frac{BD+CD}{2} : \mathit{tang.} \frac{1}{2} BC :: \mathit{cot.}\ \frac{1}{2} BC : \mathit{cot.} \frac{BD+CD}{2}$. Q. E. 1°. D.

2°. In the ſame triangle we ſhall likewiſe have, by *art.* 158, *tang.* AB : *tang.* AC : : *coſ.* CAD : *coſ.* BAD, and therefore, *componendo et dividendo*, *tang.* AB+*tang.* AC : *tang.* AB—*tang.* AC : : *coſ.* CAD +*coſ.* BAD : *coſ.* CAD—*coſ.* BAD : but, by *art.* 87, the relation of the two firſt terms is equal to that of, $\mathit{sin.}\ \overline{AB+AC} : \mathit{sin.}\ \overline{AB-AC}$, and that of the two ſecond, by *art.* 100, equal to that of, *cot.* $\frac{BAD+CAD}{2} : \mathit{tang.}\ \frac{BAD-CAD}{2}$; therefore, if we ſubſtitute theſe relations for the preceding, and put $\frac{1}{2}$ BAC for $\frac{BAD+CAD}{2}$, when the perpendicular falls within the triangle, but $\frac{1}{2}$ BAC for $\frac{BAD-CAD}{2}$, when it falls without, we ſhall get for the two caſes ; $\mathit{Sin.}\ \overline{AB+AC} : \mathit{sin.}\ \overline{AB-AC} ::$ $\mathit{cot.}\ \frac{1}{2} BAC : \mathit{tang.}\ \overline{\frac{1}{2} BAD \mp \frac{1}{2} CAD}$. Q. E. 2°. D.

SCHOLIUM.

178. The firſt part of this Theorem, it is evident, conduces to the ſolution of a ſpherical triangle whereof one ſide and the two adjacent angles are given ; by immediately reducing it into two right-angled ones, in each of which will be known one ſide and the adjacent angle. We might likewiſe uſe it for the ſame purpoſe in a right-lined triangle ; only its application becomes unneceſſary for finding the ſides of ſuch a triangle, ſeeing they may be more eaſily obtained by the well known analogy between

between the *ſines* of angles and their oppoſite ſides; and therefore, the only caſe, wherein this Theorem can be of ſervice in plane Trigonometry, is that which requires the ſegments of the baſe of a triangle to be found. For, if we ſuppoſe the baſe and two adjacent angles given; then, in order to find the ſegments thereof, we ſhall have this analogy; *As the ſine of half the ſum of the angles above the baſe is to that of their difference, ſo is half the ſaid baſe to half the difference or half the ſum of its ſegments, according as the perpendicular falls within or without the triangle.*

The ſecond part of the Theorem may be applied to a ſpherical triangle, when there are given two of its ſides and the included angle, in order to reduce it into two right-angled ones, in each of which we ſhall know the hypothenuſe and an adjacent angle. It may likewiſe be applied to the correſponding caſe of right-lined triangles, in order to find the ſegments of the vertical angle, by ſubſtituting, $\overline{AB+AC}$ and $\overline{AB-AC}$, for the *ſines* of theſe quantities; and in this caſe, the common rule will not be found much ſimpler.

Theorem III.

179. *I ſay moreover, that in any ſpherical triangle,* BAC, *we ſhall have theſe two proportions*; 1°, *as the ſine of half the ſum of any two angles is to that of half their difference, ſo is the tangent of half the ſide adjacent to theſe angles to the tangent of half the difference of their oppoſite ſides*; or, which is the ſame, *Sin.* $\frac{C+B}{2}$: *ſin.* $\frac{C-B}{2}$ :: *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{AB-AC}{2}$; and 2°, *as the coſine of half the ſum of the angles above one of the ſides is to that of half their difference, ſo is the tangent of half the adjacent ſide to that of half* *Fig.* 15.

 the

the ſum of the oppoſite ſides; that is, *Coſ.* $\frac{C+B}{2}$: *coſ.* $\frac{C-B}{2}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{AB+AC}{2}$.

DEMONSTRATION.

We proved in the laſt Theorem that, *ſin.* $\overline{C+B}$: *ſin.* $\overline{C-B}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{BD-CD}{2}$; but, $C+B = \frac{2\times\overline{C+B}}{2}$, and, $C-B=\frac{2\times\overline{C-B}}{2}$; alſo, by the formula in *art.* 71, (*ſin.* 2A = 2 *ſin.* A × *coſ.* A), *ſin.* $\overline{C+B}$ will be found = 2 *ſin.* $\frac{C+B}{2}$ × *coſ.* $\frac{C+B}{2}$, and, for the ſame reaſon, *ſin.* $\overline{C-B}$ = 2 *ſin.* $\frac{C-B}{2}$ × *coſ.* $\frac{C-B}{2}$; and therefore, the foregoing proportion will, after dividing its two firſt terms by 2, be changed into the following; *ſin.* $\frac{C+B}{2}$ × *coſ.* $\frac{C+B}{2}$: *ſin.* $\frac{C-B}{2}$ × *coſ.* $\frac{C-B}{2}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{BD-CD}{2}$. Moreover, ſince in every ſpherical triangle the *ſines* of the angles are to each other as the *ſines* of their oppoſite ſides, we ſhall have, *ſin.* C : *ſin.* B : : *ſin.* AB : *ſin.* AC, and conſequently, *componendo et dividendo*, *ſin.* C + *ſin.* B : *ſin.* C − *ſin.* B : : *ſin.* AB + *ſin.* AC : *ſin.* AB − *ſin.* AC; which proportion will, by ſubſtituting for its ſeveral terms their equals, as given in *art.* 91, 92, and 95, become, *ſin.* $\frac{C+B}{2}$ × *coſ.* $\frac{C-B}{2}$: *ſin.* $\frac{C-B}{2}$ × *coſ.* $\frac{C+B}{2}$: : *tang.* $\frac{AB+AC}{2}$: *tang.* $\frac{AB-AC}{2}$. Now, if we multiply this proportion by that which we before found, and expunge the contradictory quantities, we ſhall get, *ſin.*^{2} $\frac{C+B}{2}$: *ſin.*^{2} $\frac{C-B}{2}$: :

tang.

$tang. \frac{1}{2} BC \times tang. \frac{AB+AC}{2} : tang. \frac{BD-CD}{2} \times tang. \frac{AB-AC}{2}$; likewiſe, if we multiply the antecedents of the *alternate* proportion of the firſt Theorem by $tang. \frac{AB+AC}{2}$, and the conſequents by $tang. \frac{BD-CD}{2}$, we ſhall again get, $tang. \frac{1}{2} BC \times tang. \frac{AB+AC}{2} : tang. \frac{BD-CD}{2} \times tang. \frac{AB-AC}{2} : : tang.^2 \frac{AB+AC}{2} : tang.^2 \frac{BD-CD}{2}$; and therefore, ſince theſe two proportions have one common relation, we ſhall, by reaſoning *ex æquo*, obtain this analogy; $ſin.^2 \frac{C+B}{2} : ſin.^2 \frac{C-B}{2} :: tang.^2 \frac{AB+AC}{2} : tang.^2 \frac{BD-CD}{2}$, or, by extracting the roots, $Sin. \frac{C+B}{2} : ſin. \frac{C-B}{2} : : tang. \frac{AB+AC}{2} : tang. \frac{BD-CD}{2}$: : by *Theo.* I. $tang. \frac{1}{2} BC : tang. \frac{AB-AC}{2}$; from whence the truth of the firſt part of the Theorem manifeſtly follows.

2°. We might prove by a ſimilar calculation that, $Coſ. \frac{C+B}{2} : coſ. \frac{C-B}{2} : : tang. \frac{1}{2} BC : tang. \frac{AB+AC}{2}$: * for the whole art of this calculation conſiſts in diſ-

* Since by the firſt part of the Theorem, $coſ. \frac{C+B}{2} \times ſin. \frac{C+B}{2} : coſ. \frac{C-B}{2} \times ſin. \frac{C-B}{2} : : tang. \frac{1}{2} BC : tang. \frac{BD-CD}{2}$, and alſo, $coſ. \frac{C+B}{2} \times ſin. \frac{C-B}{2} : coſ. \frac{C-B}{2} \times ſin. \frac{C+B}{2} : : tang. \frac{AB-AC}{2} : tang. \frac{AB+AC}{2}$; therefore, by multiplying theſe

disposing the two proportions which we multiplied together in such a manner, as to obtain the squares of the *cosines* of the quantities, C+B and C—B; a thing which is attended with very little difficulty. ——And hence the truth of the famous Theorem of *Neper* relating to both these cases is clearly shewn.

SCHOLIUM.

180. It is easy to perceive, that this Theorem may be applied in order to find the sides of any spherical triangle, whereof the base and the angles above it are known. It might also be used for the same purpose in the corresponding case of plane Trigonometry, after substituting the base, and the sum or difference of the sides, for the *tangents* of these quantities.———It may likewise be observed, that the preceding demonstrations carry along with them two other Theorems not altogether unworthy our regard. The first is this; *In*

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these two proportions together, we shall have, $cos.^2 \frac{C+B}{2} : cos.^2 \frac{C-B}{2} :: tang. \frac{1}{2} BC \times tang. \frac{AB-AC}{2} : tang. \frac{AB+AC}{2} \times tang. \frac{BD-CD}{2}$: but, from *Theo.* I. we shall get, $tang. \frac{1}{2} BC \times tang. \frac{AB-AC}{2} : tang. \frac{AB+AC}{2} \times tang. \frac{BD-CD}{2} :: tang.^2 \frac{AB-AC}{2} : tang.^2 \frac{BD-CD}{2}$; and consequently, *ex æquo*, $cos.^2 \frac{C+B}{2} : cos.^2 \frac{C-B}{2} :: tang.^2 \frac{AB-AC}{2} : tang.^2 \frac{BD-CD}{2}$, or, $Cos. \frac{C+B}{2} : cos. \frac{C-B}{2} :: tang. \frac{AB-AC}{2} : tang. \frac{BD-CD}{2} :: tang. \frac{1}{2} BC : tang. \frac{AB+AC}{2}$.

any

any spherical triangle, the sine of half the sum of the angles at the base is to the sine of half their difference, as the tangent of half the sum of their opposite sides is to the tangent of half the difference of the segments of the base; and the second this; *The cosine of half the sum of the angles at the base is to the cosine of half their difference, as the tangent of half the difference of their opposite sides is to the tangent of half the difference of the segments of the base.*——————Moreover, from this Theorem, and the consideration that every right-lined triangle has the sum of its three angles equal to two right ones, these two general analogies will arise; 1°, *As the sine of half the vertical angle is to the cosine of half the difference of the other two angles, so is the base or side opposite to this angle to the sum of the other two sides*; that is to say, *Sin.* $\frac{1}{2}$ A : *cos.* $\frac{C-B}{2}$: : BC : AB+AC; and 2°, *as the cosine of half the vertical angle is to the sine of half the difference of the other two angles, so is the base to the difference of the opposite sides*; or, *Cos.* $\frac{1}{2}$ A : *sin.* $\frac{C-B}{2}$: : BC : AB—AC ——— Lastly, the two analogies, which we have obtained from the demonstration of this Theorem, will give us this double analogy for any right-lined triangle, when the segments of the base are taken into consideration; *As the sum or difference of the sides is to the difference of the segments of the base, so is the cosine or sine of half the vertical angle to the sine or cosine of half the difference of the angles above the base*; that is, AB+AC : BD—CD : : *cos.* or *sin.* $\frac{1}{2}$ A : *sin.* or *cos.* $\frac{C-B}{2}$.

THEOREM IV.

181. *In every spherical triangle*, BAC, *we shall moreover have these two analogies*; 1°, *as the sine of half* Fig. 11.

half the sum of the sides including any angle is to that of half their difference, so is the cotangent of half the included angle to the tangent of half the difference of the other two angles; or, *Sin.* $\frac{AB+AC}{2}$: *sin.* $\frac{AB-AC}{2}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{C-B}{2}$; and 2°, *as the cosine of half the sum of any two sides is to that of half their difference, so is the cotangent of half their included angle to the tangent of half the sum of their opposite angle*; that is to say, *Cos.* $\frac{AB+AC}{2}$: *cos.* $\frac{AB-AC}{2}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{C+B}{2}$.

DEMONSTRATION.

In the triangle DEF, all the parts whereof are supplemental to those in the triangle BAC, we shall have, by the preceding Theorem, *sin.* $\frac{E+D}{2}$: *sin.* $\frac{E-D}{2}$: : *tang.* $\frac{1}{2}$ DE : *tang.* $\frac{FD-FE}{2}$. Now, in order to find what this analogy will become in the triangle BAC, let a and b be two circular arcs, representing, for instance, the angles E and D, and let $2r$ be put $=180°$; then will the supplements to those angles be, $2r-a$ and $2r-b$, and half the sum of those supplements, $2r - \frac{a}{2} - \frac{b}{2}$; that is to say, the supplement to half the sum of the said angles, a and b; consequently, *sin.* $\frac{E+D}{2}$ = *sin.* $\frac{AB+AC}{2}$; and also, *sin.* $\frac{E-D}{2}$ = *sin.* $\frac{AB-AC}{2}$: lastly, since *tang.* $\frac{1}{2}$ supplement of an arc = *cot.* of half that arc, we shall have, *tang.* $\frac{1}{2}$ DE = *cot.* $\frac{1}{2}$ BAC, and, *tang.* $\frac{FD-FE}{2}$ = *tang.* $\frac{C-B}{2}$. Q. E. 1°. D.

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We might ſhew exactly in the ſame manner, that the proportion, *coſ.* $\frac{E+D}{2}$: *coſ.* $\frac{E-D}{2}$: : *tang.* $\frac{1}{2}$ DE : *tang.* $\frac{DF+FE}{2}$, would be changed into this; *Coſ.* $\frac{AB+AC}{2}$: *coſ.* $\frac{AB-AC}{2}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{C+B}{2}$. Q. E. 2°. D.

SCHOLIUM.

182. We may perceive, by inſpection only, that theſe two analogies may be applied in order to find the two remaining angles of a ſpherical triangle, wherein two of the ſides and their included angle are known. We may likewiſe perceive, that the firſt analogy, when applied to the correſponding caſe of right-lined triangles, will give the common rule; *As the ſum of the ſides is to their difference, ſo is the cotangent of half their included angle to the tangent of half the difference of the oppoſite angles.* With reſpect to the ſecond analogy, it is manifeſt, that, it cannot be applied to right-lined triangles, ſince *cot.* $\frac{1}{2}$ BAC then becomes equal to *tang.* $\frac{C+B}{2}$, becauſe in ſuch triangles the ſum of the three angles is neceſſarily equal to two right ones; which is a thing that can never prevail in ſpherical triangles, as before obſerved.——We might proceed to deduce other Theorems*, ſimilar to thoſe which we gave in the laſt Scholium; only we think it better to leave to Learners the pleaſure of finding theſe out by their own ingenuity.

* They, who would ſee a variety of other properties, whether in plane or ſpherical Trigonometry, beſides thoſe which we have thought proper to ſpecify, may have recourſe to Mr. *Emerſon*'s moſt comprehenſive and ſublime Treatiſe thereon; where we preſume they will find their curioſity abundantly ſatisfied.

TABLE

TABLE I. *For the Resolution of all the possible cases of right-angled spherical Triangles.*

N. B. The letters S and A denote the cases wherein the parts are *separated* from, or *adjacent* to, the *mean*, according to the general Theorem of *Neper*, in *art.* 147.

Given.	*Required.*	*Values of the terms required.*		*Theorems.*	*Cases wherein the Terms required are less than 90°*
Hypothenuse and one leg.	Angle opposite to the given leg.	Its *sin.* = $\frac{\textit{sin. given leg}}{\textit{sin. hypoth.}}$.	S	*Th.* 1	If the given leg be less than 90°.
	Angle adjacent to the given leg.	Its *cos.* = $\frac{\textit{tang. given leg}}{\textit{tang. hypoth.}}$.	A	*Th.* 2	If the things given be of the same affection.
	Other leg.	Its *cos.* = $\frac{\textit{cos. hypoth.}}{\textit{cos. given leg}}$.	S	*Th.* 3	Idem.
One leg and its opposite angle.	Hypothenuse.	Its *sin.* = $\frac{\textit{sin. given leg}}{\textit{sin. given ang.}}$.	S	*Th.* 1	Ambiguous.
	Other leg.	Its *sin.* = $\frac{\textit{tang. given leg}}{\textit{tang given ang.}}$.	A	*Th.* 5	Idem.
	Other angle.	Its *sin.* = $\frac{\textit{cos. given ang.}}{\textit{cos. given leg.}}$.	S	*Th.* 4	Idem.

One leg and the adjacent angle.	Hypothenuse.	Its *tang.* = $\frac{tang.\ given\ leg}{cos.\ given\ angle}$.	A	*Th.* 2.	If the things given be of the same affection.
	Other angle.	Its *cos.* = *cos. giv. leg* × *sin. giv. ang.*	S . .	*Th.* 4.	If the given leg be less than 90°.
	Other leg.	Its *tang.* = *sin. giv. leg* × *tang. giv. ang.*	A .	*Th.* 5.	If the given angle be less than 90°.
Hypothenuse and one angle.	Adjacent leg.	Its *tang.* = *tang. hyp.* × *cos. giv. ang.*	A .	*Th.* 2.	If the things given be of the same affection.
	Leg opp. to the giv. ang.	Its *sin.* = *sin. hyp.* × *sin. giv. ang.*	S . .	*Th.* 1.	If the given angle be acute.
	Other angle.	Its *tang.* = $\frac{cot.\ given\ angle}{cos.\ hypothenuse}$.	A	*Th.* 6.	If the things given be of the same affection.
The two legs.	Hypothenuse.	Its *cos.* = *rectangle cos. giv. legs.*	S . .	*Th.* 3.	If the given legs be of the same affection.
	Either of the angles.	Its *tang.* = $\frac{tang.\ opposite\ leg}{sin.\ adjacent\ leg}$.	A . . .	*Th.* 5.	If the opposite leg be less than 90°.
The two angles.	Hypothenuse.	Its *cos.* = *rect. cot. giv. angles.*	A . .	*Th.* 6.	If the angles be of the same affection.
	Either of the legs.	Its *cos.* = $\frac{cos.\ opposite\ angle}{sin.\ adjacent\ angle}$.	S . . .	*Th.* 4.	If the opposite angle be acute.

TABLE II. *For the Reſolution of oblique-angled ſpherical Triangles.*

N. B. We denote the firſt and ſecond ſegment of the angle or ſide, which is divided by a perpendicular, by I. and II. *ſeg.*

Given.	Required.		Values of the Terms required.
Two ang. and a ſide oppoſite to one of them.	The ſide opp. to the other angle.	*By the common analogy.*	*The ſin. of ang. are as the ſin. of their opp. ſides.*
	Third ſide.	*Let fall a per. upon the ſide contain'd betwixt the given ang.*	*Tang.* I. ſeg. of this ſide $=$ *coſ. adj. ang.* $\times$ *tang. giv. ſide.* *Sin.* II ſeg. $= \dfrac{\textit{ſin. I. ſeg.} \times \textit{tang. ang. adj. to given ſide}}{\textit{tang. ang. opp. to the ſaid ſide}}$.
	Third angle.	*Let fall a per. as before.*	*Cot.* I. ſeg of this ang. $=$ *coſ. giv. ſide* $\times$ *tang. adj. angle.* *Sin.* II. ſeg. $= \dfrac{\textit{ſin. I. ſeg.} \times \textit{coſ. ang. opp. to given ſide}}{\textit{coſ. angle adj. to the ſaid ſide}}$.
Two ſides and an angle oppoſite to one of them.	The angle opp. to the other ſide.	*By the com. analogy.*	*The ſin. of ſides are as the ſin. of their oppoſite angles.*
	Angle included betwixt the giv. ſides.	*Let fall a perpen. from the included angle.*	*Cot.* I. ſeg. ang. ſought. $=$ *tang. giv. ang.* $\times$ *coſ. adj. ſide.* *Coſ.* II. ſeg. $= \dfrac{\textit{coſ. I. ſeg.} \times \textit{tang. giv. ſide adj. to given ang.}}{\textit{tang. ſide opp. to the ſaid angle}}$.
	Third ſide.	*Let fall a perpen. as before.*	*Tang.* I. ſeg. of this ſide $=$ *coſ. giv. ang.* $\times$ *tang. of its adj. ſide.* *Coſ.* II. ſeg. $= \dfrac{\textit{coſ. I. ſeg.} \times \textit{coſ. ſide opp. given angle}}{\textit{coſ. ſide adj. to the ſaid angle}}$.

Two ſides and the included angle.	An ang. opp. to one of the given ſides.	*Let fall a perpen. from the third angle.*	*Tang.* I. ſeg. of div. ſide $=$ *coſ. giv. ang.* $\times$ *tang. ſide opp. ang. ſought.* *Tang.* ang. ſought $= \dfrac{\textit{tang. giv. ang.} \times \textit{ſin.}\ \text{I.}\ \textit{ſegment}}{\textit{ſin.}\ \text{II.}\ \textit{ſeg. of div. ſide}}$.
	Third ſide.	*Let fall a perpen. upon one of the giv. ſides.*	*Tang.* I. ſeg. of div. ſide $=$ *coſ. giv. ang.* $\times$ *tang. other giv. ſide.* *Coſ.* ſide ſought $= \dfrac{\textit{coſ. ſide not div.} \times \textit{coſ.}\ \text{II.}\ \textit{ſegment}}{\textit{coſ.}\ \text{I.}\ \textit{ſeg. of ſide divided}}$.
A ſide and the two adjacent angles.	A ſide opp. to one of the giv. angles.	*Let fall a perpen. upon the third ſide.*	*Cot.* I. ſeg. of div. ang. $=$ *coſ. giv. ſide* $\times$ *tang. ang. opp. ſide ſought.* *Tang.* ſide ſought $= \dfrac{\textit{tang. giv. ſide} \times \textit{coſ.}\ \text{I.}\ \textit{ſeg. div. ang.}}{\textit{coſ.}\ \text{II.}\ \textit{ſeg. of the ſaid ang.}}$.
	Third angle.	*Let fall a perpen. from one of the giv. ang.*	*Cot.* I. ſeg. div. ang. $=$ *coſ. giv. ſide* $\times$ *tang. other giv. ngle.* *Coſ.* angle ſought $= \dfrac{\textit{coſ. ang. not div.} \times \textit{ſin.}\ \text{II.}\ \textit{ſeg.}}{\textit{ſin.}\ \text{I.}\ \textit{ſeg. of div. angle}}$.
The three ſides.	*Let,* a, b, c, *be put for the three ſides, and,* s, *for their ſum; and let,* b *and* c, *be ſuppoſed to be the ſides which contain the angle ſought.*		
	An ang. by the *ſin.* or *coſ.* of its half.		*Sin.* $\frac{1}{2}$ ang. $= \dfrac{\sqrt{\textit{ſin.}\,\frac{1}{2}s-b \times \textit{ſin.}\,\frac{1}{2}s-c}}{\sqrt{\textit{ſin.}\,b \times \textit{ſin.}\,c}}$; *coſ.* $\frac{1}{2}$ ang. $= \dfrac{r\sqrt{\textit{ſin.}\,\frac{1}{2}s \times \textit{ſin.}\,\frac{1}{2}s-a}}{\sqrt{\textit{ſin.}\,b \times \textit{ſin.}\,c}}$.
The three angles.	*Let,* α, β, γ, *repreſent the three angles, the ſum whereof put* $=$ s; *and let,* β *and* γ, *be ſuppoſed adjacent to the ſide required.*		
	A ſide by the *ſin.* or *coſ.* of its half.		*Sin.* $\frac{1}{2}$ ſide $= \dfrac{r\sqrt{\textit{coſ.}\,\frac{1}{2}s \times \textit{coſ}\,\frac{1}{2}s-\alpha}}{\sqrt{\textit{ſin.}\,\beta \times \textit{ſin.}\,\gamma}}$; *coſ.* $\frac{1}{2}$ ſide $= \dfrac{r\sqrt{\textit{ſin.}\,\frac{1}{2}s-\beta \times \textit{ſin.}\,\frac{1}{2}s-\gamma}}{\sqrt{\textit{ſin.}\,\beta \times \textit{ſin.}\,\gamma}}$.

TABLE III. *For the Reſolution of all the poſſible caſes of oblique-angled ſpherical Triangles by the Analogies of* Neper, *as demonſtrated in art.* 175, &c.

Given.	*Required.*	*Values of the Terms required.*
Two ſides and the included angle.	The other two angles.	*Tang.* $\frac{1}{2}$ their diff. $=\dfrac{\textit{cot. } \frac{1}{2} \textit{ giv. ang.} \times \textit{ſin. } \frac{1}{2} \textit{ diff. of giv. ſides}}{\textit{ſin. } \frac{1}{2} \textit{ ſum of theſe ſides.}}$. *Tang.* $\frac{1}{2}$ their ſum $=\dfrac{\textit{cot. } \frac{1}{2} \textit{ giv. ang.} \times \textit{coſ. } \frac{1}{2} \textit{ diff. of giv. ſides}}{\textit{coſ. } \frac{1}{2} \textit{ ſum of the ſaid ſides}}$.
	Third ſide.	*By the com. analogy. The ſin. of angles are as the ſin. of their oppo. ſides.*
Two ſides and an oppoſite angle.	Angle oppoſite to the other known ſide.	*By the com. analogy. The ſin. of ſides are as the ſin. of their opp. ang.*
	Third angle.	*Cot.* of its half $=\dfrac{\textit{tang. } \frac{1}{2} \textit{ diff. of other two ang.} \times \textit{ſin. } \frac{1}{2} \textit{ ſum giv. ſides}}{\textit{ſin. } \frac{1}{2} \textit{ diff. of theſe ſides}} = \dfrac{\textit{tang. } \frac{1}{2} \textit{ ſum of other two ang.} \times \textit{coſ. } \frac{1}{2} \textit{ ſum giv. ſides}}{\textit{coſ. } \frac{1}{2} \textit{ diff. of the ſaid ſides}}$.
	Third ſide.	*By the common analogy.*

Two angles above one of the ſides.	The other two ſides.	*Tang*. $\frac{1}{2}$ their diff. $= \frac{\textit{tang.}\ \frac{1}{2}\ \textit{giv. ſide} \times \textit{ſin.}\ \frac{1}{2}\ \textit{diff. giv. ang.}}{\textit{ſin.}\ \frac{1}{2}\ \textit{ſum of theſe angles}}$. *Tang*. $\frac{1}{2}$ their ſum $= \frac{\textit{tang.}\ \frac{1}{2}\ \textit{giv. ſide} \times \textit{coſ.}\ \frac{1}{2}\ \textit{diff. giv. angles}}{\textit{coſ.}\ \frac{1}{2}\ \textit{ſum of the ſaid angles}}$.
	The third ang.	*By the com. analogy. The ſin. of ſides are as the ſin. of their opp. ang.*
Two angles and one of their oppoſite ſides.	Side opp. to the other giv. angle.	*By the common analogy.*
	Third ſide.	*Tang*. of its half $= \frac{\textit{tang.}\ \frac{1}{2}\ \textit{diff. given ſides} \times \textit{ſin.}\ \frac{1}{2}\ \textit{ſum oppo. angles}}{\textit{ſin.}\ \frac{1}{2}\ \textit{diff. of theſe angles}} = \frac{\textit{tang.}\ \frac{1}{2}\ \textit{ſum given ſides} \times \textit{coſ.}\ \frac{1}{2}\ \textit{ſum oppo. angles}}{\textit{coſ.}\ \frac{1}{2}\ \textit{diff. of the ſaid angles.}}$.
	Third angle.	*By the common analogy.*
The three ſides.	One of the angles.	*Let fall a perpen. upon the ſide adja. to the ang. ſought.* *Tang*. $\frac{1}{2}$ ſum or $\frac{1}{2}$ diff. of the ſeg. of the baſe. $= \frac{\textit{tang.}\ \frac{1}{2}\ \textit{ſum} \times \textit{tang.}\ \frac{1}{2}\ \textit{diff. of the ſides}}{\textit{tang.}\ \frac{1}{2}\ \textit{baſe}}$. *Coſ*. angle ſought=*tang. adja. ſeg.* × *cot. adja. ſide.*
The three angles.	One of the ſides	*Will be obtained by finding its correſpondent angle in a triangle, which hath all its parts ſupplemental to thoſe of the triangle, whoſe three angles are given.*

CHAP. III.

Containing the Graphical or Geometrical Resolution of ſpherical Triangles.

Obſervation upon the nature of theſe Solutions.

183. GEometers in general take very little notice of the *Graphical* or *Geometrical* ſolutions, and more eſpecially ſo, ſince logarithms were invented; conſidering, that the *Numerical* ſolutions are by their means rendered ſuſceptible of the greateſt ſimplicity, and that a much greater preciſion can be thence obtained, than can poſſibly be expected from conſtructions; becauſe, the preciſion in this caſe depends conjointly upon the art of the perſon who conſtructs, and the goodneſs of the inſtruments wherewith he conſtructs. We ſhould therefore have paſſed over this part entirely in ſilence, had it not been for the following conſiderations.

1°. *Geometrical* ſolutions, though not altogether exact for the reaſons juſt aſſigned, yet nevertheleſs appear ſtrictly ſo to the mind, as being founded upon the moſt approved and well-known truths of Geometry. 2°. There are ſeveral caſes, wherein all the preciſion of calculation is not abſolutely requiſite, and the facility of theſe operations very conſpicuous; as, when we would apply certain obſervations (whoſe accuracy we did not greatly regard) to the ſetting of a plan or walk according to ſome point of the compaſs; or to Dialling, in order to pave the way to a more perfect operation afterwards. 3°. There are Theories in Aſtronomy which are founded upon theſe ſolutions; and alſo,

caſes

cases of Pilotage, wherein they may be found of no inconsiderable use. They may likewise serve to conduct us in calculations, wherein we might be afraid of committing some mistake; and, if we apply the Algebraical analysis to them, we may be enabled to discover a great number of new solutions, which will not only serve to illustrate this part, but also lead us to some very elegant solutions of cases in practical Astronomy, which would prove very complicated by other methods; as we shall endeavor to prove in the sequel by their applications to various Problems. The formulæ, which we shall by this means obtain, will, if compared with the *Synthetic* solutions, point out to us some remarkable differences betwixt Synthesis and Analysis; and at the same time serve to increase the diligence of Learners, by putting them into a method of finding solutions deduced from Geometrical considerations. In the last place, it may be observed that, formulæ in general are much more simply and elegantly investigated, when the calculations are founded upon Geometrical constructions, than when upon substitutions purely Algebraical; as will easily appear to any one, who will take the trouble of comparing our solutions with those, which Mr. *De la Caille* hath given for the same formulæ.

PROBLEM I.

184. *Given two of the sides*, AB, AC, *of a spherical triangle*, BAC, *with their included angle*, A; *to find* 1°, *either of the angles above the base*, *and* 2°, *the base itself or third side*, BC. *Fig.* 17.

SOLUTION.

Upon the plane of the circle ABRD*r* let there be taken the arc AB equal to one of the given sides of

of the triangle BAC; to the centre G let the radii AG and BG be drawn, and through it the diameter rGR perpendicular to the radius AG; alſo, let there be taken on each ſide of the point A the arcs, AL, Al, equal to the given ſide AC, and through the extremities of theſe arcs the chord Ll drawn: then, at the centre G let the angle DGR be made equal to the given angle BAC; and, letting fall from the point D the perpendicular Dd upon the diameter Rr, let CH be taken a fourth proportional to the lines, rG, dD, lH; and the point C will, as we have already obſerved in *art.* 160, be the projection of the angle C. This being done, in order to find one of the angles, as B for inſtance; through the point C draw the chord fCXF, as alſo the diameter MGm, perpendicular to the radius GB, and make, $Xf : XC :: GM : Gn$; then, through the point n, thus determined, draw the line nN perpendicular to GM, and terminated by the circumference at N, and the arc MN will be the meaſure of the angle B.——The angle C might be found by an exactly ſimilar operation, by taking AC upon the plane of the circle ABRar inſtead of the arc AB. Q. E. 1°. I.

2°. The third ſide BC is manifeſtly equal to either of the arcs, BF or Bf; ſince the three arcs, Bf, BC and BF, contain all the ſame number of degrees, as being compriſed betwixt the ſame point B, and a leſſer circle perpendicular to the radius GB. Q. E. 2°. I.

Scholium I.

185. The demonſtration of this conſtruction is an evident conſequence of the nature of the projection which is here made uſe of, and already explained in *art.* 160.——But, inſtead of repeating the

the operation for obtaining the value of the angle C, we may likewise find it by the following easy method; which, in order to avoid too great an extent of the figure, we shall only barely point out. Through the extremities L and F of the arcs AL and BF, respectively equal to the sides AC and BC, draw two *tangents* to meet the radii AG and BG, produced as far as necessary; then from the points of their intersection as centres, with these *tangents* as radii, describe two circular arcs, and from the point where they cut each other draw two radii to the said centres, and the angle which is contained betwixt them will be equal to the angle C: the reason of which construction will evidently appear from this consideration; that the angle which is contained betwixt two planes is equal to that which is formed by two lines perpendicular to their common intersection; as are in this case the *tangents* of the arcs AC and BC.

Scholium II.

186. Having taken, as before, the arcs, AL, A*l*, each equal to the side AC, the projection of the angle C may be determined without making use of proportionals, in the following manner. From the point H as centre, with the radius HL, describe a semicircle L*cl*; also, at the same point make with the radius HL an angle LH*c* equal to the given angle BAC, and from the point *c*, where the radius H*c* cuts the circumference, let fall the perpendicular *c*C upon the diameter *l*L, and the point C will be the projection sought of the angle C of the triangle BAC; only observe that, if the angle at A be obtuse, the point C will fall upon H*l*; but, if acute, upon HL.——The projection of the angle B may be found with equal fa-

 cility;

cility: for if, upon the line lH, produced if neceſſary, Cx be taken $=CX$, and, from the point c to x, the line cx be drawn, the angle cxL or cxl will be equal to the angle ABC.

Now in order to conceive the reaſon of this conſtruction, we muſt have recourſe to *fig.* 16, and we ſhall therein perceive, that the angle CHL is equal to the angle BAC, becauſe the lines, CH, HL, are both perpendicular to the common interſection, AG, of the planes, GAB, GAC; and conſequently, the angle cHL equal to the angle CHL. We ſhall alſo perceive, that in the triangle CXc (right-angled at c) the angle CXc, contained between the two lines, cX, CX, perpendicular to the common interſection, GB, of the planes, BGC, BGA, is equal to the angle ABC: but, according to our method of conſtruction, the right-angled triangle cCx in *fig.* 17, is perfectly equal to the triangle CcX in *fig.* 16, and of courſe the angle at x in both equal likewiſe.

SCHOLIUM III.

187. If we conſider the 17th figure with a little attention, we ſhall find that any triangle, ABC, determines therein three other triangles, which may be regarded as its correſpondents; viz. baC, BaC and bAC; each of which hath one angle either common with or equal to one of the angles, and its other angles ſupplemental to the other two, in the triangle BAC; as alſo, two of its ſides ſupplemental to two of the ſides of the ſaid triangle, and one ſide equal, viz. that which is oppoſite to the equal angle: and as theſe triangles contain all the variations which the given parts of a Problem can poſſibly admit of, it may not be improper by way of exerciſe, and in order to underſtand more perfectly

fectly the nature of constructions, to make application of every particular case to each of them respectively.

PROBLEM II.

188. *Given two of the angles,* A, B, *of a spherical triangle,* BAC, *with their adjacent side,* AB; *to find,* 1°, *the other two sides, and* 2°, *the third angle.* *Fig.* 17.

SOLUTION.

Having taken upon the circle ARD*ar* the arc AB equal to the side given, and drawn to the centre G the radii, AG, BG, erect the lines, G*r*, GM, respectively perpendicular thereto; then take the arcs, MN, RD, respectively equal to the angles at B and A, and having drawn the *sines*, N*n*, D*d*, with the semi-axes, GB and G*n*, GA and G*d*, describe the semi-ellipses, B*nb*, A*da*; and their point of intersection, C, will be the projection of the angle C of the sperical triangle BAC: then through this point let the chords, *l*CL, *f*CF, be drawn perpendicular to the radii, GA, GB, and they will give the arcs, AL, A*l*, equal to the side AC, and the arcs, BF, B*f*, equal to the side BC. Q. E. 1°. I.

2°. The sides AC and BC being thus obtained, the angle C may be determined by either of the methods of solution given in the preceding Problem. Q. E. 2°. I.

PROBLEM III.

189. *Given two of the sides,* AB, AC, *of a spherical triangle,* BAC, *with an angle opposite to one of them; to find* 1°, *the third side, and* 2°, *the other two angles.* *Fig.* 17.

Solution I.

Let us suppose the angle given to be, B, and also that it is known of what affection the angle A is, which is included betwixt the given sides, AB, AC. Then in the first place, take upon the plane of the circle ABR*ar* an arc, AB, equal to the given side AB, and draw the radii, GA, GB: which being done, through G draw MG*m* perpendicular to the radius GB, and take G*n* thereon equal to the *cosine* of the angle B towards M, if the angle BAC be obtuse, or G*v* equal to the said *cosine* towards *m*, if acute; and with, B*b*, *nv*, as axes, describe the ellipsis B*nbv*. In the next place, take upon the circumference of the said circle on each side of the point A the arcs, AL, A*l*, equal to the other given side AC; then draw the chord L*l*, and through the two points, C, C′, where it cuts the ellipsis B*nbv*, draw the lines, *f*CX, φC′*x*′, perpendicular to the radius GB; and they will give the arcs, B*f*, Bφ, equal to the third side BC, according as the opposite angle A is supposed to be obtuse or acute. Q. E. 1°. I.

2°. Now in order to find the angle A; having described the semi-circle *l*KL upon the chord L*l* as diameter, and drawn the ordinates, C*c*, C′*c*′, draw also from the point H the lines, H*c*, H*c*′, and they will give the angles, LH*c*, LH*c*′, equal to angle A, according as it is obtuse, or acute—The angle C may be found by the construction in *art.* 185. Q. E. 2°. I.

Solution II.

If we suppose the sides AC and BC with the angle B opposite to the side AC to be given, we may solve the Problem by a different (though not more com-

complicated than the preceding) method, as follows. First, assume any point, B, upon the circumference of the circle ABR*ar* for the vertex of the angle B, and through this point and the centre G draw the diameter BG*b*; perpendicular to which draw the diameter MG*m*, and take *n*G thereon equal to the *cosine* of the given angle B; then taking on each side of the point B the arcs, BF, B*f*, equal to the given side BC, and drawing the chord F*f*, make, MG : *n*G : : *f*X : XC; and it is evident, that the point C (determined by this proportion upon the plane of the circle AR*ar*) will be the projection of the angle C. In the next place, through the centre G and point C draw the radius GCP, and upon the line GC as diameter describe a circle; then, from the centre G with the radius GH, equal to the *cosine* of the given side AC, describe an arc of a circle, and through the points, H *h*, where it cuts the circle described upon GC, draw the lines, GHA, G*h*α; and they will give, AB, or αB, for the side required, according as the unknown angle opposite to the given side BC is supposed to be obtuse or acute—We may likewise find the point C by a construction exactly similar to that given in *art.* 186: for, if we describe a semi-circle upon the chord *f* F, draw through the point X a radius which may make with *f* X an angle equal to the angle B, and from the extremity thereof let fall a perpendicular upon the said chord *f* F; the point C will by this means be determined independent of proportionals.— The angles A and C may be determined as in the last Problem. Q. E. I.

Problem IV.

190. *Given two of the angles*, A, B, *of a spherical triangle*, BAC, *with a side*, AC, *opposite to one of them; to find*, 1°, *the side opposite to the other angle*; 2°, *the third side, and* 3°, *the third angle.* *Fig.* 17.

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SOLUTION.

Through any point, A, upon the circumference of the circle ABR*ar* let the diameter AG*a* be drawn, and perpendicular thereto the diameter *r*GR. This done, on each ſide of the ſaid point let the arcs, AL, A*l*, be taken equal to the given ſide AC, and the chord L*l* drawn; upon which as diameter let the ſemi-circle *l*KL be deſcribed, and from the point H the radii, H*c*, H*c*′, drawn ſo as to make with LH an angle equal to the given angle A, according as it is obtuſe or acute: then from the points, *c*, *c*′, let fall the perpendiculars, *c*C, *c*′C′, and they will determine the points, C, C′, for the projections of the angle, C, or C′. In the next place, from the points, *c*, *c*′, let the lines, *cx*, *c*′*x*′, be drawn ſo as to make with the ordinates, C*c*′, C′*c*′, the angles, C*cx*, C′*c*′*x*′, equal to the complement of the given angle B; this done, let the line GCP be drawn, and upon CG a circle, GHC*h*, deſcribed; after which, from the point C as centre with a radius equal to C*x* (equal, by *art*. 186, to CX, whoſe poſition will determine the ſides required) deſcribe a circular arc interſecting the former circle in two points X; then the ſide AB will be obtained by drawing the radius GXB; and a chord, *f*CXF, drawn through the points, X, C, will give the arcs, B*f*, BF, equal to the third ſide BC. ——The third angle may ſtill be determined by the conſtruction in *art*. 185. Q. E. I.

SCHOLIUM.

191. If the angles A and B be both acute, the ſide AB (and thence BC likewiſe) will be determined by means of the point X, which is found in the quadrant AG*r*. It is equally manifeſt that, if

we

we ſuppoſe a circle to be deſcribed upon C′G, and the line C′x′ to be inſcribed therein, we ſhall obtain two other ſolutions of this Problem; viz. one, when the angle A in the triangle BAC is acute, and B obtuſe; and the other, when the angles at A and B are both acute, and that at C obtuſe. — We have omitted drawing the ſeveral lines neceſſary to the demonſtration, through fear of rendering the figure too confuſed.

Problem V.

192. *Given the three ſides of a ſpherical triangle,* BAC; *to find one of its angles.* *Fig.* 17.

Solution I.

Having taken upon the plane of the circle AR*ar* the arc AB equal to one of the given ſides, and drawn through the extremities thereof, A, B, the radii, AG, BG; take on each ſide of the point A the arcs, AL, A*l*, equal to the ſide AC, and alſo the arcs, B*f*, BF, equal to the third ſide BC; then draw the chords, L*l*, F*f*, and their point of interſection, C, will be the projection of the angle C. This done, draw through the centre G the diameters, *r*GR, *m*GM, perpendicular to the radii, GA, GB; then find a fourth proportional, G*n*, to the lines, *f*X, CX and MG, and through the point *n* (which will by this means be determined upon MG) draw the perpendicular *n*N, terminated by the circumference at N, and the arc MN will be equal to the meaſure of the angle B. In like manner, find a fourth proportional, G*d*, to the lines, *l*H, CH and *r*G, and through the point *d* (thus determined upon *r*G) draw the perpendicular *d*D, and it will give the arc RD equal to the angle at A.

A.—The angle C might be found by an exactly similar construction, by taking the side AC upon the circumference of the circle ARar instead of AB. Q. E. I.

SOLUTION II.

If we would solve the Problem without making use of proportionals, it may be easily effected thus. Having found, as before, the point C by the intersection of the right lines, *l*L, F*f*, upon L*l* as diameter describe the semi-circle LK*l*, and through the point C draw C*c* perpendicular to the said diameter; then from the point *c* to H draw the line *c*H, and it will give the angle *c*HL equal to the angle at A. In the next place, take C*x*=CX, and draw the line *cx*, and the angle *cx*C will be equal to that at B.—The angle C might be found by a similar construction. Q. E. I.

PROBLEM VI.

193. *Given the three angles of a spherical triangle; to find one of its sides.*

SOLUTION.

Fig. 11. Make a triangle DEF, which may have its three sides supplemental to the angles of the given triangle; then, by one of the constructions in the preceding Problem, find the several angles of this triangle, and they will be the supplements to the sides required. Q. E. I.

General Scholium for the foregoing solutions.

194. It is manifest that the six last Problems contain the Geometrical solutions of all the cases of oblique-angled spherical triangles; but the constructions

structions may be equally (only with much greater simplicity) applied to right-angled triangles. We have omitted giving the demonstration of every particular solution, seeing they may be all deduced as so many Corollaries from the general constructi-on of the orthographic projections already explained in *art.* 160, and laid down in *fig.* 16; the perfect understanding of which being once supposed, all the ideas necessary to a complete demonstration of these solutions will follow of course.——We shall now subjoin some select Problems relative to this kind of projection, as being that whereof the greatest and most frequent use is made in Astronomy.

PROBLEM VII.

195. *Given the ellipsis which is the orthographic projection of a great circle, to find upon the plane of projection the appearance of the poles of this circle; and contrarily, having the projected poles of a great circle, to find the ellipsis which is the projection of the said circle.*

SOLUTION.

Let us suppose the right line A*a* to represent the plane of the great circle of the sphere, upon which the orthographic projection of all the points of the spherical surface is made, and that P*p*, perpendicular to GL, is the axis of the circle to be projected, which will consequently be represented by GL; then it is evident, that the circle, AB*ab*, which passes through the poles, P, *p*, of the circle to be projected, will be perpendicular to the plane of projection. This being premised, if we let fall the perpendiculars, Pϖ, G*g*, it will easily appear that, ϖ will be the projection of the pole P, and *g* that of the extremity G of the diameter GL of the circle to be projected: but, since the angle GCP is a right *Fig.* 19.

one, the angle GCA will be the complement of the angle aCP, which denotes the elevation of the pole above the plane of projection; and therefore, it follows that, the distance Cϖ of the projection of the pole P of a great circle C is equal to the *cosine* of the elevation of the said pole above the plane of projection, or, which is the same, to the *sine* of the elevation of the circle to be projected above the plane of projection; and that half the less axis of the ellipsis, which is the projection of this circle, is equal to the *sine* of the elevation of the pole above the plane of projection. Q. E. I.

Problem VIII.

196. *To find the dimensions of the ellipsis, which is the orthographic projection of a lesser circle of the sphere.*

SOLUTION.

Fig. 19. Let the line MON represent a diameter of the lesser circle, whereof the orthographic projection is required; also, let this line be perpendicular to the axis Pp, and the lesser circle, which it represents, will be parallel to the circle GL, whereof Pp is the axis: then from the points, M, O and N, let fall the perpendiculars, Mm, Oo and Nn, upon the line Aa; and, it is manifest that, the point o will be the centre of the ellipsis, which is to be the projection of the lesser circle MN parallel to GL, and mn the less axis thereof. This being premised, on account of the similar triangles, CPϖ, COo, we shall have, CP : CO : : Cϖ : Co; that is to say, *As radius is to the sine of the distance of the lesser circle from the great circle to which it is parallel, so is the cosine of the elevation of the pole of the said great circle above the plane of projection to the distance* Co *of the centre* o *of the ellipse to be described from the centre* C. The similar triangles, GCg, MOR, likewise give,

give, GC : Cg : : MO : OR or $om=on$; that is, *As the ſine total is to the ſine of the elevation of the pole of the circle* GL *above the plane of projection, ſo is the ſine of the diſtance of the leſſer circle from the ſaid pole to half the conjugate axis of the ellipſis which is to be deſcribed*, and whereof the tranſverſe axis is manifeſtly equal to MN.——The point *m* might alſo be found by taking C*m* equal to the *ſine* of the difference of the arcs, MG, BG; and conſequently the ellipſis ſought, deſcribed. Q. E. I.

Application of the two laſt Problems to the projections uſed in the Theory of Eclipſes.

197. The Earth being ſituate at a prodigious diſtance from the Sun, all the rays which come from thence hither may be conſidered as parallel; and if we imagine a plane perpendicular to the ray which iſſues from the centre of the ſun to that of the earth, this plane (which will cut the ſphere into two equal parts) will repreſent the diſk of the earth as appearing from the ſun; alſo ſeparate the enlightened from the darkened hemiſphere, and laſtly have all the points of the hemiſphere expoſed to the ſun projected orthographically upon it. This being premiſed, if, P, *p*, repreſent the poles of the world, the line GL will denote the equator; the little circle MN will be a parallel whereof GM is the Geographical latitude; the circle AB*ab* will be the circle which Aſtronomers call the univerſal meridian; the line BC will repreſent the ray which comes from the centre of the ſun to that of the earth, and conſequently the arc BG be the ſun's declination; the line AC*a* will be a line perpendicular to the plane of the circle B*cb*, which has the axis of the ecliptic projected upon it at the ſolſtices, and makes with the ſaid axis an angle equal to the obliquity of the ecliptic at the equinoxes; AC*a*

will represent the circle upon which all the points of the earth seen from the sun are orthographically projected, and lastly, aP be the elevation of the pole above the illuminated disk, and manifestly equal to the sun's declination. Now if, after these definitions, we would describe the ellipsis which is the projection of the parallel MN, it is evident that, the Problem will be exactly the same with that in the last article; except that the analogies, which we there gave for finding the centre *o* and half the less axis of the ellipsis to be described, will here be expressed thus: 1°. *As radius is to the sine of the geographical latitude of the parallel, so is the cosine of the sun's declination to the distance* Co; and 2°. *As the sine total is to the sine of the sun's declination, so is the cosine of the geographical latitude of the parallel to half the conjugate axis of the ellipsis which is the projection of the said parallel**.

198. As these graphical solutions very frequently occur in Astronomy, but particularly in the determination of the transits of the planets over the sun's disk, we shall here give the method of delineating upon the illuminated disk the parallel of a place, whereof the latitude is known, according to the foregoing principles.

Fig. 20. Let AG*a*Q be the circle upon which the projection is to be made. Through any point, A, taken upon the circumference thereof, draw the diameter A*a*, and on each side of the said point take the arcs, AP, A*p*, equal to the sun's declination for the time when the projection is required: then draw the chord P*p*, and the point ϖ, where it cuts the diameter A*a*, will evidently be the projection of the

* For further satisfaction see Mr. *De la Lande*'s Astronomie, p. 700, &c.

the pole P. In the next place, draw to the point P the radius CP; perpendicular to which draw the diameter ÆCQ, and it will represent the equator: then, from the points, Q, Æ, take the arcs, Ql, ÆL, equal to the latitude of the place (towards A if it be of the same denomination with the sun's declination), and draw the chord Ll, cutting the diameter Pp' in the point K; through which draw Hb parallel to Pp, and it will give O for the centre of the ellipsis to be described. This done, take CN equal to the *sine* of the difference of the arcs, AP, ÆL, and ON will be half the less axis of the required ellipsis, whereof Mm equal to Ll is the greater axis: therefore, if we take On = ON, and upon the axes, Mm, Nn, describe an ellipsis MNmn, it will be the projection of the parallel given for the day when the sun's declination is equal to the arc AP, as also touch the circle AGaQ in two points, G, I; through which if we draw the line GI (perpendicular to the radius CA), and reduce the arc GNI into hours, minutes and seconds, we shall get the length of the day for the said parallel at the time when the declination is equal to the arc AP.

199. The projection of the equator upon the plane AGaQ may likewise be easily found; for if, from the extremities, Æ, Q, of the diameter ÆQ, which represents the equator, we let fall the perpendiculars, ÆS, Qs, and upon the axes, Bb, Ss, describe the ellipsis BSb s, it will shew the projection of the terrestrial equator as seen from the sun at the time when the declination is equal to the arc AP.

200. We may also find upon the same plane the projection of the ecliptic, as well as of the equinoctial points. But in order to this it must be observed,

ſerved, that, as the axis of the equator makes always with that of the ecliptic an angle equal to the obliquity of the ecliptic, the pole P will deſcribe round the axis of the ecliptic the circumference of a leſſer circle, which will have its radius equal to the *ſine* of this obliquity, and be the baſe of a cone whoſe perpendicular height is equal to the *coſine* of the ſaid obliquity, taken upon the axis of the ecliptic: therefore, if upon the line C♑, as diameter, we deſcribe a ſemi-circle CV♑, and inſcribe therein a line CV equal to the *coſine* of the obliquity of the ecliptic, and then draw the line VC*r*, we ſhall by this means manifeſtly obtain the axis of this great circle. Moreover, ſince the ſun is always in the plane of the ecliptic, it is evident that this circle muſt be repreſented by a right line; conſequently, if ECT be drawn perpendicularly to its axis RC*r*, this line will be the projection of the ſaid circle, and the points, ♈, ♎, where it cuts the ellipſis BS*bs*, thoſe of the equinoctial points. In the next place, if through the point ♈ we draw the ordinate *f*♈F perpendicularly to the diameter B*b*, it will determine the arc *a*F equal to the ſun's right aſcenſion; and, if we alſo draw through the ſame point the line ♈D perpendicularly to the diameter ECT, the arc *r*D will be the meaſure of the ſun's longitude. —We might proceed to find with equal facility the appearance of the ſolſticial points, both upon the equator and ecliptic; only we think it unneceſſary to inſiſt any longer upon this head, ſince it is ſufficient for our purpoſe to have ſhewn how eaſily the orthographic projections may be applied to the principal Problems of ſpherical Aſtronomy. We ſhall obſerve however, that ſeveral Aſtronomers ſeem not to entertain a ſufficiently juſt idea of the nature of theſe conſtructions; for, whilſt ſome reject

ject them entirely, as destitute of a requisite exactness, others there are on the contrary, who place too much dependance upon them: the use, which we think ought to be made of them, is that, which is generally made of all Geometrical figures; viz. to conduct the mind in the research of numerical formulæ by the application of Algebra, or the general rules of Trigonometry: for which reason, and that the Theory of this kind of projections may be the better illustrated, we shall annex a few Corollaries upon the foregoing Problems.

COROLLARY I.

201. It follows from the last Problem, that in order to determine upon any ellipsis, which is the projection of a great circle, an arc which may contain any number of degrees, and also commence at a certain point upon this ellipsis; we need only let fall from the given point a perpendicular upon its transverse axis, and produce it to the circumference of the circle described upon this axis as diameter; then, from the point where the circumference of the said circle is cut by this perpendicular, set off an arc equal to the number of degrees given, and afterwards from the extremity thereof let fall another ordinate upon the transverse axis of the ellipsis, and there will by this means be determined upon the periphery of the ellipsis an arc containing the number of degrees assigned.

COROLLARY II.

202. If the arc is to contain 90°: having let fall from the given point a perpendicular upon the transverse axis of the ellipsis, we need only take on the said axis, on the other side of the centre, a line equal

equal to the *sine* of the arc comprehended betwixt that extremity of the axis which is nearest to the given point, and the perpendicular or ordinate which passeth through this point. For example, if we would determine upon the ellipsis AC*da* an arc of 90°, commencing at the point C; we need only take the line Gλ equal to the *sine* *l*H of the arc A*l*=AC, terminated by the ordinate which passes through the point given, and then at the point λ, thus obtained, erect the perpendicular $\lambda l'$, and it will give upon the said ellipsis the arc C*l'* equal to 90°. In like manner, if, upon half the greater axis G*b* of the ellipsis BC*b*, we take the line Gφ equal to the *sine* FX of the arc BF=BC, and erect the perpendicular $\varphi f'$, we shall get upon this ellipsis the arc Cf' = 90°.

Fig. 21.

Corollary III.

203. From what has been already said, it will be very easy to find upon the plane of the circle ABR*ar* the measure of the angle C or BCA; for, as every spherical angle is measured by the arc of a great circle comprised betwixt its sides, and described from its vertex as pole, it is manifest that the whole difficulty will consist in describing the ellipsis Q*l'q*, which is the projection of the great circle whose pole is at C; the method whereof is as follows. Through the point C and centre G draw the diameter PCG*p*, and perpendicular thereto through the same points the lines, G'C*g'*, QG*q*; then take upon G*p* a part, GS, equal to G'C, which is the *sine* of the elevation of the point C above the plane of projection, and this line will be half the less axis of the ellipsis which is the projection of the great circle whose pole is at C: then the points, *l'*, *f'*, where this ellipsis cuts the ellipses, A*da*, B*nb*,

B*nb*, will determine the arc $f'l'$, whose correspondent, $\beta\gamma$, upon the circle ABR*ar*, contained betwixt the ordinates, $\gamma\lambda$, $\beta\phi$, will be the measure of the angle at C.

COROLLARY IV.

204. But, in order to find the measure of the angle C, it may be easily perceived that, it is not absolutely necessary to describe the ellipsis which is the projection of the great circle whereof this angle is the pole: for, having determined, as before, the points, l', f', upon the ellipses, AC*a*, B*nb*, if from these points we let fall upon the diameter QG*q*, perpendicular to GCP, the ordinates, $l'\lambda$, $f\phi$; their prolongations to the circumference of the circle AR*ar* will determine the arc $\beta\gamma$ equal to the measure of the angle C.

PROBLEM IX.

205. *To find upon the surface of the circle* ABR*ar*; *1°, the projection of the arc* Aδ *let fall from the angle* A *perpendicularly upon the opposite side* BC; *2°, the values of the segments*, Bδ, Cδ, *of the side* BC; *3°, the segments of the angle* BAC, *and 4°, the value of the perpendicular* Aδ. *Fig.* 21.

SOLUTION.

Since the arc, whose projection we seek, is part of a great circle of the sphere passing through the point A, it is manifest that the line AG*a* will be the transverse axis of the ellipsis which is the projection thereof: moreover, since this circle is to be perpendicular to the side BC, it must necessarily pass through the pole of this arc, and of consequence the ellipsis required also through the projection of the said pole: therefore, if we take upon the radius

Gm a part GZ=Nn, the point Z will, by *art.* 195, be the pole of the arc BC, as well as one of the points of the ellipsis to be described; then, if through this point Z we draw a line Zζ perpendicularly to the diameter Aa, and produce it till it meets the circumference of the circle ARar in a point θ, and likewise take upon RG a line, Gρ, a fourth proportional to the lines, $\theta\zeta$, ζZ and RG; the ellipsis A$\delta\varrho a$ described upon the semi-axes, GA, Gρ, will be the required projection of the circle perpendicular to BC. Q. E. 1°. I.

2°. If through the point δ, where this ellipsis cuts the side BC, $\nu\delta\omega\mu$ be drawn perpendicularly to the radius GB, the arcs, $f\mu$, Bμ, will be the segments of the side BC. Q. E. 2°. I.

3°. If at the point ρ, we raise $\rho\sigma$ perpendicularly to the radius GR, the arcs, Rσ, Dσ, will be the measures of the segments, BAδ, CAδ, of the angle BAC. Q. E. 3°. I.

4°. Lastly, if through the point δ we draw the ordinate $\varepsilon\delta\eta$ perpendicularly to the radius GA, the arc Aη will be the value of the perpendicular Aδ. Q. E. 4°. I.

SCHOLIUM I.

206. If it was required to determine the projection of the circle, which should divide the angle BAC into two equal parts, it might be effected by an exactly similar construction; by taking upon the radius GR a line, Gk, equal to the *cosine* of half the angle BAC, and describing an ellipsis upon GA and Gk as semi-axes.——If the perpendicular was supposed to fall from the angle C upon the side AB, produced as far as necessary, the Problem might then be solved by means of the construction given in *art.* 203. Lastly, it is manifest, from the construction of this Problem, that the arc Rθ is equal

equal to the arc Aϑ, perpendicular to the ſide BC, ſince the arcs, AR, ηθ, each 90°, have the part ηR common.

SCHOLIUM II.

207. This conſtruction naturally leads us to a very ſimple and very elegant demonſtration of one of the famous analogies of *Neper*, nearly reſembling that which he himſelf hath given us in his Treatiſe entitled, "*Logarithmorum mirifici Canonis Conſtructio.*"——Having determined the ſeveral parts ſpecified in the foregoing Problem by the method above given, at the extremity, A, of the diameter A*a* erect the indefinite perpendicular AΛ; then from the point B as pole, with a circular radius equal to the arc BC, deſcribe a leſſer circle upon the ſurface of the ſphere, and it will manifeſtly paſs through the points, L, C, *c* and *l*: moreover, through the points, C, *c*, draw the right lines, FC*f*, ϖω*co* parallel to the *tangent* AK; and it is evident that the arc A*f* will be equal to the arc AC, as alſo the arc A*o* equal to the arc A*c*, which is the difference of the ſegments, Aϑ, ϑC, formed upon the ſaid arc AC by the perpendicular Bϑ. This being premiſed, if we in the next place draw from the point *a* through the points, L, *l*, *f* and *o*, the right lines, *a*LΛ, *al*λ, *af*K and *aok*, and produce them to the *tangent* AΛ; it will eaſily appear that the four points, Λ, λ, K and *k*, will be in the circumference of a circle deſcribed upon the plane repreſented by AΛ, and touching the ſphere at A, and that this circle is the baſe of an oblique cone whoſe vertex is at *a*. To demonſtrate which, it will be ſufficient to prove that the ſection of a cone, which has for its baſe the circle deſcribed upon the diameter L*l*, by the plane AΛ, is antiparallel (ſubcontrary) to the ſaid baſe; ſince it is ſhewn Fig. 22.

in several Treatises upon Conics that such a section is circular. Now the angle $A\Lambda a$, being formed by a *tangent* and *chord*, will be measured by half the difference of the arcs, AFa, AfL, comprised between its sides: but, $AL = AI$, and therefore, $AFa - AfL = AFa - AFI = aI = aL$; whence, as the half of this last arc is exactly the measure of the angle alL, the triangles, alL, $a\Lambda\lambda$, will be similar: consequently, the cone will be cut sub-contrarily to its base, and of course the circle described upon lL projected into a circle upon the plane $A\Lambda$. This proved, the interior *secants* will lastly give, $AK : A\Lambda :: A\lambda : Ak$; that is, *As the tangent of half the base*, AC, *is to the tangent of half the sum of the sides*, AB *and* BC, *so is the tangent of half the difference of these sides to the tangent of half the difference of the segments of the base.*

In order to be convinced of the identity of these two proportions, it must be observed that, by regarding the diameter Aa as radius, the lines, AK, $A\Lambda$, $A\lambda$ and Ak, will be the *tangents* of the angles, AaK, $Aa\Lambda$, $Aa\lambda$ and Aak, at the point a; and that, as these angles have their vertices in the circumference of the circle ALa, they will be measured by the halves of the arcs contained betwixt their sides: but, the arc Af is, by construction, equal to the base AC of the spherical triangle BAC; the arc AL to the sum of the sides AB and BC; Al to the difference of these sides, and Ao to the difference of the segments of the base, since the arcs, Ac, Ao, are contained betwixt the parallel planes, $A\Lambda$, ϖo, and consequently equal.

Scholium III.

208. From the preceding demonstration it may be collected that two kinds of projections have been

been made use of; the one *orthographic* of all the parts of the triangle upon the plane of the circle AL*a*; and the other upon the plane represented by A*A*, which is called the *polar* projection, because the eye was supposed to be placed at the point *a*, considered as one of the poles of the sphere. Now *these* projections have a great affinity with the *stereographic* ones (whereof very frequent use is made with respect to *Maps*), and differs from them only in this; that in the *latter*, the eye may be supposed at *any* point upon the surface of the sphere, and the projection of the opposite hemisphere made upon the plane of a great circle, produced as far as necessary, and perpendicular to the radius drawn from the centre of the sphere to the place of the eye upon the sphere's surface. The pre-eminence of these projections arises from hence, that they always give similar figures to those which are required to be projected, and consequently represent upon a plane the Map of a Country by figures similar to those which are formed upon the surface of the globe: which important property is a necessary consequence of this, that the cones of projections are always cut in a sub-contrary manner by the plane of projection. They who would thoroughly understand this subject, may have recourse to the works of Father *Tacquet*, or to the Optics of Father *Aiguillon*——We shall now, in the last place, subjoin the following Problem, since it furnishes us with an extremely easy solution of most of the cases of spherical Trigonometry, as also with a new sort of graphical solution.

PROBLEM X.

209. *Given the three sides of a spherical triangle,* BAC; *to find one of the angles thereof, by developing its parts.*

SOLU-

SOLUTION.

Fig. 23. Upon any circle, BAC, take the three arcs, AB, AC and BC′, reſpectively equal to the three ſides of the triangle whoſe angles are required; and having drawn the radius GA (if it be the angle A which we would find), draw thereto the *tangent* *d*AD, terminated at *d* and D by the prolongations of the radii GC and GB: then upon the radius GC′, produced as far as neceſſary, take the line G*d*′ equal to the *ſecant* G*d* of the ſide AC; and from the points, D, A, as centres, with the radii, D*d*′, A*d*, deſcribe two circular arcs interſecting each other in the point δ, and they will give the angle δAD equal to the angle A of the triangle BAC.—— We might find by a ſimilar conſtruction the other two angles of this triangle.

This ſolution, it is evident, carries with it its own demonſtration; for, the lines, AD, A*d* or Aδ, being the *tangents* of the arcs, AB, AC, and by conſtruction perpendicular to the common interſection, AG, of the planes, BAG, CAG, the angle formed between them, when terminated by the line Dδ = D*d*′, will manifeſtly be equal to the angle A in the triangle BAC†. Q. E. I. and D.

COROLLARY I.

210. This conſtruction, it is very plain, might be made uſe of in order to find the ſides of a triangle whereof the three angles are given, by applying it to a triangle whoſe ſeveral parts are ſupplemental to thoſe of the ſaid triangle (*art.* 130).

COROL-

† Should not this explanation appear ſufficiently clear and ſatisfactory, conſult the firſt Theorem of *Simpſon*'s Trigonometry, p. 23.

COROLLARY II.

211. It is equally apparent that, this construction may be likewise applied to the solution of a triangle, whereof two sides and the included angle are known. For, having taken the arcs, AB, AC, respectively equal to the sides containing the given angle, and drawn the *tangents*, AD, Ad, terminated by the radii, GB, GC, produced as far as necessary; we need only make the angle DAδ equal to the angle given, take Aδ equal to the *tangent* Ad, and draw Dδ; then, from the points G and D as centres, with the radii Gd'=Gd and Dd' =Dδ, describe two circular arcs, and afterwards from the point of their intersection, d', draw to the centre G the line d'G, and the arc BC′ will by this means be determined equal to the third side; from whence the other angles may be easily obtained.

CHAP.

CHAP. IV.

Containing the Analytical or Algebraical Resolution of spherical Triangles.

ADVERTISEMENT.

ALL the calculations, which we ſhall have to make in this Chapter, being founded upon the conſtructions given in the preceding one, it will be abſolutely neceſſary to underſtand *them* perfectly before we proceed any further. However, that theſe calculations may not be rendered too difficult for Learners by frequent changes of the ſigns, we ſhall uſually apply them to the 17th *fig.* the ſeveral parts whereof we ſhall denote as in the following Table: but, if any terms happen to change their ſigns when the formulæ by this means derived are applied to other triangles, we are from thence to conclude that the angles, which were before ſuppoſed acute or obtuſe, become then obtuſe or acute; or that the ſame changes are made with reſpect to the ſides.

T A B L E.

Let the *ſine total* be put $= r$, and the parts of the triangle BAC expreſſed thus;

Sin.

Sin. AB$=$BK$=a$.	*Cos.* BAC$=$G$d=q$.
Cos. AB$=$GK$=b$.	*Sin.* ACB$=h=\frac{ap}{c}=\frac{am}{f}$.
Sin BC$=$FX or fX$=c$.	*Cos.* ACB$=k=\sqrt{\frac{r^2c^2-a^2p^2}{c^2}}$
Cos. BC$=$GX$=d$.	*Tang.* AB $=\frac{ar}{b}$.
Sin. AC$=$LH or lH$=f$.	*Tang.* BC $=\frac{r}{d}$.
Cos. AC$=$GH$=g$.	*Tang.* AC $=\frac{fr}{g}$.
Sin. ABC$=$N$n=m$.	*Tang.* B $=\frac{mr}{n}$.
Cos. ABC$=$G$n=n$.	*Tang.* A $=\frac{pr}{q}$.
Sin. BAC$=$D$d=p$.	*Tang.* C $=\frac{hr}{k}$.

Then the conſtructions in *art.* 160 will give the following values of the lines, CX, Cf, &c.

$CX=\frac{cn}{r}$.	$CH=\frac{fq}{r}$.
$Cf=c-\frac{cn}{r}$.	$CL=f+\frac{fq}{r}$.
$CF=c+\frac{cn}{r}$.	$Cl=f-\frac{fq}{r}$.

For greater expedition and certainty, and to prevent the trouble of recurring inceſſantly to this Table in order to find the analytical expreſſions of the quantities it contains, it will not be improper for the Reader to tranſcribe the whole upon a piece of looſe paper, and carry the ſame along with him as he proceeds in the ſolution of the ſeveral Problems.

PROBLEM I.

212. *To find the relation betwixt the ſines of the*

angles and those of the sides of any spherical triangle.

SOLUTION.

From the intersection of the chords, fF, lL, we shall have, $CF \times Cf = CL \times Cl$, or, which amounts to the same,* $\overline{fX}^2 - \overline{CX}^2 = \overline{lH}^2 - \overline{CH}^2$; and, if
Fig. 17. for these quantities their algebraical values be substituted, we shall get, $c^2 - \frac{c^2 n^2}{r^2} = f^2 - \frac{f^2 q^2}{r^2}$, or, $c^2 r^2 - c^2 n^2 = f^2 r^2 - f^2 q^2$; which, after putting m^2 in the place of $r^2 - n^2$, and p^2 for $r^2 - q^2$, become, $c^2 m^2 = f^2 p^2$: then, if we extract the square roots of the members of this equation, and put these roots into proportion, we shall find, $c : f :: p : m$; that is, *Sin.* BC : *sin.* AC :: *sin.* BAC : *sin.* ABC: from whence it is manifest that in any spherical triangle, *The sines of the angles are to each other as those of their opposite sides.* Q. E. I.

PROBLEM II.

213. *Given two sides and their included angle; to find* 1°, *either of the other angles, and* 2°, *the third side.*

SOLUTION.

Let the sides, AB, AC, with the included angle
Fig. 17. BAC be the parts given, and let it be first required to find the angle B.———On account of the similar triangles, GKB, GHO, we shall have, GK : KB :: GH : HO, or, $b : a :: g : \frac{ag}{b}$; also, by the construction given in *art.* 160 we shall have, Gr : Gd :: lH : CH, or, $r : q :: f : \frac{fq}{r}$; therefore, CO or CH

* This will easily appear from the 5th Proposition of the 2d Book of *Euclid*.

$CH+HO=\frac{fq}{r}+\frac{ag}{b}$. Moreover, it is evident that the angle OCX in the right-angled triangle CXO is equal to the angle AGB; wherefore, we ſhall get from ſimilarity of triangles, R : *coſ.* AB : : CO : CX, and conſequently, $CX=\frac{bfq}{rr}+\frac{ag}{r}$: laſtly, ſince the lines, fX, CX, MG and nG, are proportional by conſtruction, we ſhall have, fX : CX : : MG : nG, or analytically, c ($=\frac{fp}{m}$ by *art.* 212) : $\frac{bfq}{rr}+\frac{ag}{r}$: : r : n; from whence we get, $\frac{nfp}{m}=\frac{bfq}{r}+ag$, and therefore, $\frac{m}{n}=\frac{r^2fp}{bfq+agr}=$ (by dividing the ſecond member of the equation by f) $\frac{r^2p}{bq+\frac{agr}{f}}$; that is to ſay, $Tang.\,B=\frac{RR\times sin.\,A}{sin.\,AB\times cot.\,AC+cos.\,AB\times cos.\,A}$ Q. E. 1°. I.

2°. In order to obtain the third ſide BC, we need only regard the *ſine* of this ſide as unknown, and put c in the place of $\frac{fp}{m}$ in the equation $\frac{nfp}{m}-ag=\frac{bfq}{r}$, and we ſhall get, $c=\frac{ag}{n}+\frac{bfq}{rn}$; that is, *Sin.* $BC=\frac{cos.\,AC\times sin.\,AB}{cos.\,B}+\frac{cos.\,AB\times cos.\,A\times sin.\,AC}{R\times cos.\,B}$. Q. E. 2°. D.

COROLLARY.

214. It is evident, that if we ſuppoſe different ſides with the included angle given, and purſue a ſimilar proceſs to that whereby we found the angle B, we ſhall get the following formulæ;

$$Tang.\ B = \frac{R^2 \times \sin.\ A}{\sin.\ AB \times cot.\ AC + \cos.\ AB \times \cos.\ A} = \frac{R^2 \times \sin.\ C}{\sin.\ BC \times cot.\ AC + \cos.\ BC \times \cos.\ C}.$$

$$Tang.\ C = \frac{R^2 \times \sin.\ A}{\sin.\ AC \times cot.\ AB + \cos.\ AC \times \cos.\ A} = \frac{R^2 \times \sin.\ B}{\sin.\ BC \times cot.\ AB + \cos.\ BC \times \cos.\ B}.$$

$$Tang.\ A = \frac{R^2 \times \sin.\ C}{\sin.\ AC \times cot.\ BC + \cos.\ AC \times \cos.\ C} = \frac{R^2 \times \sin.\ B}{\sin.\ AB \times cot.\ BC + \cos.\ AB \times \cos.\ B}.$$

But, here it muſt be obſerved that, ſome of the terms of theſe formulæ will have their ſigns changed, according to the different combinations of acute or obtuſe angles; ſo that, whenever we would apply them to any particular caſe, we muſt have reſpect to this conſideration. For inſtance, if we would obtain the value of the angle B of the triangle BAC in *fig.* 18, from having the ſides, AB, AC, and their included angle A given; we ſhall find, as in *fig.* 17, that, HO $= \frac{ag}{b}$, and CH $= \frac{fq}{r}$; but HO is equal to HO—CH, and therefore the *Tang*. of the angle B becomes $= \frac{R^2 \times \sin.\ A}{\sin\ AB \times cot.\ AC - \cos.\ AB \times \cos.\ A}$ —— In like manner the term *coſ*. AC×*coſ*. A would become negative in the formula for the *tangent* of the angle C, &c.

PROBLEM III.

215. *Suppoſing the ſame things given as in the preceding Problem; it is required to find the third ſide without regard to the angles adjacent thereto.*

SOLU-

SOLUTION.

For facility of calculation, we will ſuppoſe the ſides AB and BC with the included angle B to be the parts given: then, it is manifeſt that, all we have to do is to find the value of the *ſine* lH, or *coſine* GH, of the ſide AC. Now in order to do this, we ſhall get from the right-angled triangles, GKB, CXO, *coſ.* AB : *ſin.* AB : : CX : XO, or, $b : a : : \frac{cn}{r} : \frac{acn}{br}$; whence, GO or $GX + XO = d + \frac{acn}{br}$: likewiſe, from the right-angled triangles, GKB, GHO, we ſhall get, R : *coſ.* AB : : GO : GH; or, algebraically, $r : b : : d + \frac{acn}{br} : \frac{bdr + acn}{rr} =$ *Coſ.* AC. Q. E. I.

COROLLARY I.

216. Therefore, in general, when in any ſpherical triangle two of the ſides and the included angle are given, we ſhall have the following formulæ for finding the third ſide;

$$Cos.\,AC = \frac{\pm cos.B \times sin.AB \times sin.BC + R \times cos.AB \times cos.BC}{RR}.$$

$$Cos.\,AB = \frac{\pm cos.C \times sin.AC \times sin.BC + R \times cos.AC \times cos.BC}{RR}.$$

$$Cos.\,BC = \frac{\pm cos.A \times sin.AB \times sin.AC + R \times cos.AB \times cos.AC}{RR}.$$

COROLLARY II.

217. From theſe formulæ it will follow that, if V be called the *verſed ſine* of any angle C, we ſhall have, $V = \frac{R^2 \times cos.\,\overline{BC - AC} - cos.\,AB}{sin.\,AC \times sin\,BC}$. For let R — V be put in the place of *coſ.* C in the expreſſion of *coſ.* AB found in the laſt Corollary, and we ſhall get,

get, $cos.\ AB \times RR = cos.\ AC \times cos.\ BC \times R + sin.\ AC \times sin.\ BC \times R - sin.\ AC \times sin.\ BC \times V$; but, by *art.* 24, $cos.\ AC \times cos.\ BC + sin.\ AC \times sin.\ BC = R \times cos.\ \overline{BC - AC}$; therefore, by the ſubſtitution of this value and the common methods of reduction, we ſhall find that, $V = \frac{R^2 \times cos.\ \overline{BC - AC} - cos.\ AB}{sin.\ AC \times sin.\ BC} = \frac{2R \times cos.\ \overline{BC - AC} - cos.\ AB}{cos.\ \overline{BC - AC} - cos.\ \overline{BC + AC}}$; by ſubſtituting the value of $sin.\ AC \times sin.\ BC$, deduced from what hath been demonſtrated in *art.* 26.

COROLLARY III.

218. If, in the numerator of the firſt of the above values of V, we ſubſtitute for $cos.\ \overline{BC - AC} - cos.\ AB$ what may be deduced from the demonſtration in *art.* 57 to be equal to it, there will ariſe this expreſſion, $V = \frac{2R \times sin.\ \overline{\frac{1}{2} AB + \frac{1}{2} BC - \frac{1}{2} AC} \times sin.\ \overline{\frac{1}{2} AB + \frac{1}{2} AC - \frac{1}{2} BC}}{sin.\ AC \times sin.\ BC}$; and, ſince we have ſhewn, in *art.* 22, that the *verſed-ſine* of an angle $= \frac{2\ sin.^2\ \frac{1}{2}\ angle}{R}$, we ſhall get by this laſt ſubſtitution, $sin.^2\ \frac{1}{2} C = \frac{R^2 \times sin.\ \frac{AB + BC - AC}{2} \times sin.\ \frac{AB + AC - BC}{2}}{sin.\ AC \times sin.\ BC}$; and, by extracting the ſquare roots, $Sin.\ \frac{1}{2} C =$

$$\frac{R \times \sqrt{sin.\ \frac{AB + BC + AC}{2} - AC \times sin.\ \frac{AB + AC + BC}{2} - BC}}{\sqrt{sin.\ AC \times sin.\ BC}};$$

which is the ſame formula with that already given in the ſecond Chapter for finding any angle of a triangle whereof the three ſides are known, and ſhews us the uſe of the algebraical analyſis in the diſcovery of different formulæ.

SCHO-

SCHOLIUM.

219. Supposing still that we knew two of the sides of a triangle with their included angle, as the sides, AB, AC, with the included angle A of the triangle BAC, and would find the third side BC by means of its sine, we should get, $c = \frac{\sqrt{b^2f^2q^2 + 2abfgqr + a^2g^2r^2 + p^2f^2r^2}}{r^2}$*; an expression manifestly too complicated to be ever used in the business of calculations.

Problem IV.

220. *Given one side and the angles above it; to find* 1°, *either of the other sides, and* 2°, *the third angle.*

SOLUTION.

Let AB be the side given; A, B, the angles above this side, and let AC be one of the sides required; the *sine* or *cosine* of which must consequently be regarded as unknown. Then we shall in the first place have, by *art.* 112, fX or *sin.* BC $= \frac{pf}{m}$: in the next place, we shall get an expression for HO by this analogy, *cos.* AB : *sin.* AB : : GH *Fig.* 17 *and* 18.

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* Since, by *Prob.* I. $\overline{fX}^2 - \overline{CX}^2 = \overline{lH}^2 - \overline{CH}^2$, and, by *Prob.* II. $CX = \frac{bfq + agr}{r^2} \therefore c^2 - \frac{b^2f^2q^2 + 2abfgqr + a^2g^2r^2}{r^4}$ will be $= \frac{f^2r^2 - f^2q^2}{r^2} = \frac{\overline{r^2 - q^2} \times f^2}{r^2} = \frac{p^2 \times f^2}{r^2} = \frac{p^2 f^2 r^2}{r^2}$; and consequently, $c = \frac{\sqrt{b^2f^2q^2 + 2abfgqr + a^2g^2r^2 + p^2f^2r^2}}{r^2}$.

GH : HO, or, $b : a :: g : \frac{ag}{b} =$ H O : we have likewise, CH $= \frac{fq}{r}$, and consequently, CO=HO $\pm$CH$=\frac{ag}{b} \pm \frac{fq}{r}$: moreover, from the right-angled triangles, GKB, CXO, we shall get, R : *cos.* AB : : CO : CX, or, by substituting the analytical values of this proportion, $r : b :: \frac{ag}{b} \pm \frac{fq}{r} : \frac{ag}{r} \pm \frac{bfq}{rr} =$ CX: lastly, since the lines, MG, nG, fX and CX, are proportional, we shall have this analogy, R : *cos.* B : : *sin.* BC or fX : CX, or, algebraically, $r : n ::$ $\frac{fp}{m} : \frac{ag}{r} \pm \frac{bfq}{rr}$; whence we deduce, $\frac{nfp}{m} = ag \pm \frac{bfq}{r}$, and by reduction, $\frac{f}{g} = \frac{arr}{\frac{n}{m}p \mp bq}$; or, by substituting the values of the letters, *Tang.* AC $=$ $\frac{RR \times sin.AB}{cot.B \times sin.A \mp cos.AB \times cos.A}$: where the sign — takes place when the angle A is obtuse, and the sign + when it is acute. We might find in like manner that, *Tang.* AB$= \frac{RR \times sin.AC}{cot.C \times sin.A \mp cos.AC \times cos.A}$, and, *Tang.* BC $= \frac{RR \times sin.AB}{cot.A \times sin.B + cos.AB \times cos.B}$. Q. E. 1°. I.

2°. In order to find the third angle C, after having found either of the sides, let the equation $\frac{nfp}{m} = ag \pm \frac{bfq}{r}$ be resumed, and from thence let there be deduced the value of $\frac{am}{f}$; and we shall get, $\frac{am}{f} = \frac{rnp \mp bmq}{gr}$; or, *Sin.* C $= \frac{sin.A \times cos.B}{cos.AC} \mp$ $\frac{cos.AB \times cos.A \times sin.B}{R \times cos.AC}$. Q. E. 2° I.

PRO-

PROBLEM V.

221. *Supposing the same things given as in the preceding Problem; it is required to find the third angle in expressions of the sides and angles given.*

SOLUTION.

That we may the more easily obtain a formula for the part sought, we will suppose the two angles A and C with the side AC to be known, and the angle B to be that required; the *sine*, *m*, or *cosine*, *n*, of which must consequently be considered as unknown. Then we shall first have, by the common proportion between the *sines* of angles and those of their opposite sides, *sin.* AB or BK $= \frac{fh}{m}$, *Fig.* 17 *and* 18.

and, *sin.* BC or $fX = \frac{pf}{m}$: in the next place, we shall get from the triangles, GKB, GHO, *cos.* AB : *sin.* AB :: GH : HO, or, $\frac{\sqrt{r^2 m^2 - f^2 h^2}}{m} : \frac{fh}{m} :: g : \frac{gfh}{\sqrt{r^2 m^2 - f^2 h^2}}$: we have also by construction, R : *cos.* A :: *sin.* AC : CH, or, $r : q :: f : \frac{fq}{r} =$ CH; and consequently, CO $=$ HO $\pm$ CH $= \frac{gfh}{\sqrt{r^2 m^2 - f^2 h^2}} \pm \frac{fq}{r}$: likewise, from the right-angled triangles, GKB, CXO, we shall get, R : *cos.* AB. :: CO : CX, or, in species, $r : \frac{\sqrt{r^2 m^2 - f^2 h^2}}{m} :: \frac{fgh}{\sqrt{r^2 m^2 - f^2 h^2}} \pm \frac{fq}{r} : \frac{f}{mr} \pm \frac{fq\sqrt{r^2 m^2 - f^2 h^2}}{mrr} =$ C X: moreover,

 since

ſince, MG, nG, fX and CX, are proportional, we ſhall have, R : *coſ.* B :: fX : CX, or, $r : n ::$ $\frac{pf}{m} : \frac{fgh}{mr} \pm \frac{fq\sqrt{r^2m^2 - f^2h^2}}{mrr}$; and therefore, $\frac{pfn}{m}$ $= \frac{fgh}{m} \pm \frac{fq\sqrt{r^2m^2 - f^2h^2}}{mr}$. Now, if we divide each member of this equation by $\frac{f}{m}$, and take away the radical, we ſhall have $p^2n^2 - 2pghn + g^2h^2 = q^2m^2 - \frac{f^2h^2q^2}{r^2}$: then, if we put $r^2 - n^2$ in the place of m^2, r^2 for $p^2 + q^2$, and divide the whole by r^2, we ſhall get, $n^2 - \frac{2pghn}{r^2} = q^2 - \frac{g^2h^2}{r^2} - \frac{f^2h^2q^2}{r^4}$; which equation, after completing the ſquare according to the common rules for quadratics, and different reductions and ſubſtitutions, becomes, $n^2 - \frac{2pghn}{r^2} + \frac{p^2g^2h^2}{r^4} = \frac{q^2k^2}{r^2}$; whence we deduce, $n = \frac{pgh}{r^2} \pm \frac{qk}{r}$; that is, *Coſ.* B $= \frac{\text{ſin. A} \times \text{ſin. C} \times \text{coſ. AC}}{\text{RR}} \pm \frac{\text{coſ. A} \times \text{coſ. C}}{\text{R}}$. —— We might find by ſimilar proceſſes that, *Coſ.* C $= \frac{\text{ſin. A} \times \text{ſin. B} \times \text{coſ. AB}}{\text{RR}} \pm \frac{\text{coſ. A} \times \text{coſ. B}}{\text{R}}$, and that, *Coſ.* A $= \frac{\text{ſin. B} \times \text{ſin. C} \times \text{coſ. BC}}{\text{RR}} \pm \frac{\text{coſ. B} \times \text{coſ. C}}{\text{R.}}$. Q. E. I.

REMARK.

We might have avoided all the calculations which we have made in order to obtain the ſolution of this Problem, by applying the formula in *art.* 215 to a triangle whoſe parts ſhould be ſeverally

rally the supplements to those in the triangle BAC, and afterwards deducing from it by proper substitutions the value of the *cosine* of the angle required; only we judged it better to find this value by a direct process, that we might render the analytical solutions as easy and familiar as possible.

COROLLARY.

222. From the formula, *tang.* AC $= \frac{RR \times sin.\ AB}{cot.\ B \times sin.\ A + cos.\ AB \times cos.\ A}$, given in *art.* 220, it will be easy to find the following values of the *cot.* of this side, by inverting the fraction, and putting *cot.* A B for $\frac{cos.\ AB}{sin.\ AB}$; viz. *Cot.* A C $= \frac{cot.\ B \times sin.\ A}{sin.\ AB} \mp cot.\ AB \times cos.\ A = \frac{cot.\ B \times sin.\ C}{sin.\ BC} \mp cot.\ BC \times cos.\ C.$

PROBLEM VI.

223. *Given two sides and an angle opposite to one of them; to find* 1°. *the angle included between these sides, and* 2°, *the side adjacent to the given angle.*

SOLUTION.

Let the sides, AB, AC, and the angle B opposite to the side AC, be supposed to be given, and let the angle A be that which is required. Then let the equation, $\frac{nfp}{m} = ag + \frac{bfq}{r}$, in *art.* 220 be resumed, and the value of, p, q, or $\frac{p}{q}$ (which are the *sine*, *cosine* and *tangent* of the angle A) thence deduced, and the thing will be done. Now, if for greater ease we put, $\frac{m}{n}$, the *tangent* of the an- *Fig.* 17

gle B $= t$, and, $\frac{f}{g}$, the *tangent* of the ſide AC $=$ s, we ſhall firſt get, $p - \frac{at}{s} = \frac{btq}{rr}$, or, by putting $\sqrt{r^2 - p^2}$ in the place of q, $p - \frac{at}{s} = \frac{bt\sqrt{r^2-p^2}}{rr}$; from whence we deduce this quadratic equation, $p^2 - \frac{2atr^4p}{s\times\overline{r^4+b^2t^2}} = \frac{b^2t^2r^2s^2 - a^2t^2r^4}{s^2\times\overline{r^4+b^2t^2}}$; and, by the common method of calculation, $p = \frac{atr^4 \pm btr\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\times\overline{r^4+b^2t^2}}$. We might find in like manner, by ſubſtituting for p inſtead of q, the value of the *coſine* of this angle, or, $q = \frac{-abt^2r^2 \pm r^3\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\times\overline{r^4+b^2t^2}}$; and conſequently, from theſe values of p and q, an analytical expreſſion for the *tangent* of this angle. Q. E. 1°. I.

2°. Suppoſing ſtill that the ſides and angle given are, AB, AC and B, and that we would find the ſide BC, or, which will amount to the ſame, the *ſine* fX or *coſine* GX of this ſide. If in the equation, $\frac{pfn}{m} = ag + \frac{bfq}{r}$, before given, we ſubſtitute for the unknown quantities, p and q, their values, $\frac{cm}{f}$ and $\frac{1}{f}\sqrt{f^2r^2 - c^2m^2}$ (wherein c muſt be regarded as unknown), we ſhall get, $cn - ag = \frac{b}{r}\sqrt{f^2r^2 - c^2m^2}$; and thence, by the well known rules for ſolving quadratics, $c = \frac{agnr^2 \pm br\sqrt{b^2f^2m^2 + f^2n^2r^2 - a^2g^2m^2}}{b^2m^2 + n^2r^2}$. We might

might likewise find from the above equation the value of the *cosine* of BC; but, as the expression, which we should by this means obtain, would be considerably too complicated, it will be necessary to inquire whether a more simple one cannot be obtained by a farther application of Algebra to *fig.* 17. For this end, the rectangled triangles, GKB, GXτ, will give, *cos.* AB : R :: GX : Gτ, or, $b : r :: d : \frac{dr}{b}$; whence, τH = GH — Gτ = $g - \frac{dr}{b}$: the same triangles will likewise give, *cos.* AB : *sin.* AB :: GX : τX, or, $b : a :: d : \frac{ad}{b}$: also, from the right-angled triangles, GKB, CHτ, we shall get, *sin.* AB : R :: τH : Cτ, or, $a : r :: g - \frac{dr}{b} : \frac{gr}{a} - \frac{drr}{ab}$: and consequently, C$\tau$ + τX or CX $= \frac{gr}{a} - \frac{drr}{ab} + \frac{ad}{b} = \frac{gr - bd}{a}$: moreover, on account of the ordinates to the circle and ellipsis, we shall have, GM : Gn :: fX : CX, or, in species, $r : n :: \sqrt{rr - dd} : \frac{gr - bd}{a}$; from whence we immediately get, $an\sqrt{rr - dd} = grr - bdr$; and therefore, $d = \frac{r^3 bg \pm anr\sqrt{b^2 r^2 + a^2 n^2 - g^2 r^2}}{b^2 r^2 + a^2 n^2}$. We might find in like manner, by solving another quadratic equation, *Sin.* BC or C $= \frac{agnrr \pm br^2\sqrt{a^2 n^2 + b^2 r^2 - g^2 r^2}}{a^2 n^2 + b^2 r^2}$; and from these two values of d and c deduce that of the *tangent* of the same side likewise. Q. E. 2°. I.

PRO-

PROBLEM VII.

224. *Given two angles and one of their opposite sides; to find 1°, the adjacent side; 2°, the third angle, and 3°, the third side.*

SOLUTION.

Fig. 17. Let A and B be the two known angles, and the side given AC, opposite to the angle B, and let the side AB be that required. Then, in the equation, $\frac{bfq}{r} + ag = \frac{fnp}{m}$ (which contains all the given parts of the Problem, together with the *sine*, a, and *cosine*, b, of the unknown side AB), we need only find the value of either of the quantities a and b, considered as unknown, and the thing will be done. But, that we may render the calculation as simple as possible, we will in the first place divide each member of the above equation by f; then put t for the *cotangent*, $\frac{g}{f}$, of the side AC; s for the *cotangent*, $\frac{n}{m}$, of the angle B, and make the radius equal to unity; and the said equation will become, $bq + at = ps$, or, $q\sqrt{r^2 - a^2} = ps - at$; from whence, by raising the whole to the square, and solving a quadratic, we shall easily get, *Sin.* AB or $a = \frac{pst \pm q\sqrt{r^2t^2 + q^2r^2 - p^2s^2}}{t^2 + q^2}$. We might find in like manner, by substituting for a, and solving another quadratic equation, *Cos.* AB or $b = \frac{pqs \pm t\sqrt{r^2t^2 + q^2r^2 - p^2s^2}}{t^2 + q^2}$; and from these two values of the side AB an algebraical expression for

for the *tangent* thereof might also be easily obtained. Q. E. 1°. I.

2°. For the angle C. Since the analogy between the *sines* of angles and those of their opposite sides gives, *sin.* C $= \frac{am}{f}$; therefore, in order to obtain an expression for the *sine* of this angle in given parts of the Problem, it is manifest that, we need only multiply the *sine*, *a*, of the side AB by $\frac{m}{f}$, and there will arise, *Sin* C $=$ $\frac{pst \pm q\sqrt{r^2t^2 + q^2r^2 - p^2s^2}}{t^2 + q^2} \times \frac{m}{f}$. Q. E. 1°. I.

3°. Lastly, to find the third side BC, it is evident that, we need only make use of the common analogy, and it will give, *Sin.* BC $= \frac{sin. AC \times sin. A}{sin. B}$. Q. E. 3°. I.

PROBLEM VIII.

225. *Given the three sides of a spherical triangle; to find one of its angles.*

SOLUTION.

Let the equation, CX $= \frac{gr - bd}{a}$, found in the second part of the sixth Problem, be resumed: Fig. 17.
then the construction of the figure gives us, *f* X : CX : : MG : *n*G, or, $c : \frac{gr - bd}{a} :: r : cos. B = \frac{grr - bdr}{ac}$; and therefore, by substituting the value of each letter, *Cos.* B will be found $= \frac{cos. AC \times RR - cos. AB \times cos. BC \times R}{sin. AB \times sin. BC}$. —— We might find in like manner that, *Cos.* A $= \frac{cos. BC \times RR - cos. AB \times cos. AC \times R}{sin. AB \times sin. AC}$, and, *Cos.* C $= \frac{cos. AB \times RR - cos. AC \times cos. BC \times R}{sin. AC \times sin. BC}$.

PRO-

PROBLEM IX.

226. *Given the three angles of a spherical triangle; to find one of its sides.*

SOLUTION.

By still pursuing the analysis arising from the construction of *fig.* 17. it would be easy to obtain an equation containing the *cosine* of the side required, exprest by, or combined with, the given parts
Fig. 11. of the Problem: but in the triangle DEF, whose parts are severally the supplements to those in the triangle BAC, and whose three sides are given, we shall have, $Cos.\,D = \frac{cos.\,EF \times RR - cos.\,DF \times cos.\,DE \times R}{sin.\,DF \times sin.\,DE}$; and therefore, since the angle D is the supplement to the side AB, and arcs which are the supplements to each other have the same *sine* and *cosine*, we shall get, by making such substitutions as the figure requires, $Cos.\,AB = \frac{cos.\,C \times RR - cos.\,A \times cos.\,B \times R}{sin.\,A \times sin.\,B}$. —— We might find in like manner that, $Cos.\,AC = \frac{cos.\,B \times RR - cos.\,A \times cos.\,C \times R}{sin.\,A \times sin.\,C}$, and that, $Cos.\,BC = \frac{cos.\,A \times RR - cos.\,B \times cos.\,C \times R}{sin.\,B \times sin.\,C}$. Q. E. I.

SCHOLIUM I.

227. If the algebraical solutions of the sixth and seventh Problems be compared with those which we have demonstrated synthetically in the second Chapter, we shall be struck with surprise at the prodigious difference which will appear between the one and the other; nay, we shall be almost induced to believe that there is some impropriety in

in the algebraical analysis, when we see how complicated some of the expressions are which arise from its use. This difference, however, deserves to be examined with the greatest attention, and may discover to us some important truths, very useful in their applications: for which reason, we shall observe that, when the triangle BAC is reduced into two other right-angled ones in order to obtain its several parrs, if we would find the angle A, by supposing the sides, AB, AC, and the angle B opposite to one of these sides, known; the operations laid down in the general Table for oblique-angled triangles give us the value of this angle by parts; viz. the *cotangent* of BAD in the first place, and the *cosine* of CAD in the second; whilst the algebraical analysis supposes no such division of the said angle, but discovers its whole *sine* or *cosine* by one operation: and therefore, from this different manner of proceeding in the solution of the Problem, it is, that the difference of the solutions ariseth: which, nevertheless, is not real, but only apparent; as the following process will put beyond a possibility of doubt. Let x be put for the *sine* of the angle BAD; then the formula, *cot.* I. seg. of the ang. BAC = *tang. given ang.* × *cos. adj. side*, will give, $\frac{r\sqrt{rr-xx}}{x} = \frac{bt}{r}$, by preserving the substitutions used in *Prob.* VI: from whence we get, x or *sin.* BAD $= \frac{r^3}{\sqrt{r^4+b^2t^2}}$; and consequently, *cos.* BAD $= \frac{btr}{\sqrt{r^4+b^2t^2}}$. The second formula of the same article in the same Table gives, *cos.* CAD $= \frac{atrr}{s\sqrt{r^4+b^2t^2}}$; and consequently, *sin.* CAD $= \frac{r\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\sqrt{r^4+b^2t^2}}$: *Fig.* 15.

U therefore,

PROBLEM IX.

226. *Given the three angles of a spherical triangle; to find one of its sides.*

SOLUTION.

By still pursuing the analysis arising from the construction of *fig.* 17. it would be easy to obtain an equation containing the *cosine* of the side required, exprest by, or combined with, the given parts
Fig. 11. of the Problem: but in the triangle DEF, whose parts are severally the supplements to those in the triangle BAC, and whose three sides are given, we shall have, $Cos. D = \frac{cos. EF \times RR - cos. DF \times cos. DE \times R}{sin. DF \times sin. DE}$; and therefore, since the angle D is the supplement to the side AB, and arcs which are the supplements to each other have the same *sine* and *cosine*, we shall get, by making such substitutions as the figure requires, $Cos. AB = \frac{cos. C \times RR - cos. A \times cos. B \times R}{sin. A \times sin. B}$. —— We might find in like manner that, $Cos. AC = \frac{cos. B \times RR - cos. A \times cos. C \times R}{sin. A \times sin. C}$, and that, $Cos. BC = \frac{cos. A \times RR - cos. B \times cos. C \times R}{sin. B \times sin. C}$. Q. E. I.

SCHOLIUM I.

227. If the algebraical solutions of the sixth and seventh Problems be compared with those which we have demonstrated synthetically in the second Chapter, we shall be struck with surprise at the prodigious difference which will appear between the one and the other; nay, we shall be almost induced to believe that there is some impropriety in

in the algebraical analyſis, when we ſee how complicated ſome of the expreſſions are which ariſe from its uſe. This difference, however, deſerves to be examined with the greateſt attention, and may diſcover to us ſome important truths, very uſeful in their applications: for which reaſon, we ſhall obſerve that, when the triangle BAC is reduced into two other right-angled ones in order to obtain its ſeveral parrs, if we would find the angle A, by ſuppoſing the ſides, AB, AC, and the angle B oppoſite to one of theſe ſides, known; the operations laid down in the general Table for oblique-angled triangles give us the value of this angle by parts; viz. the *cotangent* of BAD in the firſt place, and the *coſine* of CAD in the ſecond; whilſt the algebraical analyſis ſuppoſes no ſuch diviſion of the ſaid angle, but diſcovers its whole *ſine* or *coſine* by one operation: and therefore, from this different manner of proceeding in the ſolution of the Problem, it is, that the difference of the ſolutions ariſeth: which, nevertheleſs, is not real, but only apparent; as the following proceſs will put beyond a poſſibility of doubt. Let x be put for the *ſine* of the angle BAD; then the formula, *cot.* I. ſeg. of the ang. BAC = *tang. given ang.* × *coſ. adj. ſide*, will give, $\frac{r\sqrt{rr-xx}}{x}=\frac{bt}{r}$, by preſerving the ſubſtitutions uſed in *Prob.* VI: from whence we get, x or *ſin.* BAD $=\frac{r^3}{\sqrt{r^4+b^2t^2}}$; and conſequently, *coſ.* BAD $=\frac{btr}{\sqrt{r^4+b^2t^2}}$. The ſecond formula of the ſame article in the ſame Table gives, *coſ.* CAD $=\frac{atrr}{s\sqrt{r^4+b^2t^2}}$; and conſequently, *ſin.* CAD $=\frac{r\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\sqrt{r^4+b^2t^2}}$: Fig. 15.

therefore, by means of these values, and since BAC $=$ BAD $\pm$ CAD, it will be easy to find the *sine* of the whole of this angle from the formula in *art.* 23, $fin.\ \overline{A \pm B} = \frac{fin.A \times cof.B \pm fin.B \times cof.A}{R}$ (supposing BAD $=$ A and CAD $=$ B); viz. $p = \frac{atr^4 \pm btr\sqrt{r^4s^2 + b^2s^2t^2 - a^2t^2r^2}}{s \times \overline{r^4 + b^2t^2}}$; which is exactly the same equation with that we found in *Prob.* VI. by solving an equation of the second degree. Hence we may perceive, how these two kinds of solutions correspond in reality, though apparently quite different; as also, with what facility equations of the second degree, which contain very complicated radicals, may be solved by two analogies of spherical Trigonometry.

SCHOLIUM II. *for right-angled spherical Triangles.*

228. But there is yet another difference which may be observed to subsist betwixt the synthetic and analytical solutions of the Problems of spherical Trigonometry: for, the former owe the simplicity of their formulæ to this; that we begin with finding values for the most simple cases, and afterwards refer to them for the values of the more difficult and complex ones: whereas in the latter, on the contrary, we suppose the solutions of general cases to be equally simple with those of particular ones; and likewise, that all particular cases are contained in the general formulæ, and may be deduced from them with the greatest facility; as, for instance, the several formulæ which relate to right-angled spherical triangles, by considering that the *sine* of a right angle is equal to the radius, but its *cosine* nothing. Thus, from the formula in *art.* 214, $tang.\ B = \frac{fin.A \times RR}{fin.AB \times cot.AC \pm cof.AB \times cof.A}$, we shall get,

get, by ſuppoſing the angle A 90°, *tang.* B $= \frac{tang\,AC \times R}{fin.\,AB}$: in like manner, the formula in *art.* 216, $cof.\,BC = \frac{R \times cof.\,AB \times cof.\,AC \pm cof.\,A \times fin.\,AB \times fin.\,AC}{RR}$, will give, $cof.\,BC = \frac{cof.\,AB \times cof.\,AC}{R}$. The angle A being ſtill ſuppoſed right, the formulæ in *art.* 220 will give, $tang.\,AC = \frac{fin.\,AB \times tang.\,B}{R}$; $tang.\,AB = \frac{fin.\,AC \times tang.\,C}{R}$, and, $tang.\,BC = \frac{tang.\,AB \times R}{cof.\,B}$; and the formulæ in *art.* 221, $cof.\,B = \frac{fin.\,C \times cof.\,AC}{R}$, and, $cof.\,C = \frac{fin.\,B \times cof.\,AB}{R}$. From the value of p found in *art.* 223 we ſhall get, by ſuppoſing the angle B right, $fin.\,A = \frac{R\sqrt{tang.^2\,AC - tang.^2\,AB}}{tang.\,AC}$* : the formula for the *coſine* of A, upon a ſuppoſition that the angle B is ſtill right, will be immediately reduced to, $q = \frac{ar^2}{sb}$; becauſe the radical in this caſe vaniſhes: therefore, in any ſpherical triangle, right-angled at B, we ſhall have, R × *coſ.* A = *tang.* AB × *cot.* AC; and, right-angled at A, R × *coſ.* B = *tang.* AB × *cot.* BC. Moreover, from the formula inveſtigated in the ſecond part of the ſame

U 2 article,

*For if in the equation, $p = \frac{atr^4 + btr^4\sqrt{r^4s^2 + b^2t^2s^2 - a^2t^2r^2}}{s \times r^4 + b^2t^2}$, the value of t be reſtored, we ſhall eaſily find, $p = \frac{amnr^4 + bmr\sqrt{n^2r^4s^2 + b^2m^2s^2 - a^2m^2r^2}}{n^2sr^4 + sb^2m^2}$; which by ſuppoſing $n = 0$ becomes, $p = \frac{bm^2r\sqrt{b^2s^2 - a^2r^2}}{sb^2m^2} = \frac{r\sqrt{b^2s^2 - a^2r^2}}{bs} = \sqrt{\frac{b^2s^2 - a^2r^2}{s^2b^2}} = r\sqrt{1 - \frac{a^2r^2}{s^2b^2}} = \frac{R\sqrt{Tang.\,AC^2 - Tang.^2\,AB}}{Tang.\,AC}$.

article, $d = \frac{r^3 bg \pm \sqrt{\&c.}}{b^2 r^2 + a\ n^2}$, we ſhall eaſily deduce, ſince n is $= 0$, $d = \frac{rg}{b}$, or, *coſ.* BC $= \frac{R \times coſ.\,AC}{coſ.\,AB}$: conſequently, in a triangle right-angled at A, *coſ.* AC will be $= \frac{R \times coſ.\,BC}{coſ.\,AB}$, and, *ſin.* AC $=$ $\frac{R\sqrt{coſ.^2 AB - coſ.^2 BC}}{coſ.\,AB}$. Again, the firſt formula in *art.* 224 gives, R $\times$ *ſin.* AB $=$ *tang.* AC $\times$ *cot.* B; becauſe the radical, being multiplied by $q = 0$, vaniſhes: but in the ſecond formula, on the contrary, the radical only remains, and we get this equation; $b = \frac{r\sqrt{t^2 - s^2}}{t}$, or, *coſ.* AB $= \frac{R\sqrt{cot.^2 AC - cot.^2 B}}{cot.\,AC.}$. Laſtly, when we know (beſides the right-angle) the two angles above the hypothenuſe, the formulæ in *art.* 226 will give; *coſ.* AB $= \frac{coſ.\,C \times R}{ſin.\,B}$; *coſ.* AC $= \frac{coſ.\,B \times R}{ſin.\,C}$, and, *coſ.* BC $\times$ R $=$ *cot.* B $\times$ *cot.* C: and hence we may readily perceive, that it had been eaſy to arrive at the General Theorem of *Neper*, as well by algebraical proceſſes, as ſynthetic conſiderations.——In order to make the generality of the preceding formulæ ſtill farther appear, we might apply them to different Problems in plain Trigonometry; but, as this is a thing very ſimple in itſelf, we ſhall leave it to the exerciſe of Learners, and add, in the next place, ſome conſiderations upon the conſtructions in *art.* 204 and 205.

PROBLEM X.

229. *Suppoſe a perpendicular be let fall from any* Fig. 21. *angle of a ſpherical triangle,* BAC, *upon the oppoſite ſide, produced if neceſſary; it is required to find in ſines, coſines, tangents or cotangents;* 1°, *the relations of the ſegments*

segments of the base to their adjacent angles; 2°, *the relations of these segments to their corresponding sides*; 3°, *the relations of the segments of the vertical angle to their adjacent sides, and* 4°, *the relations of the same segments to the angles above the base.*

SOLUTION.

Let us suppose a perpendicular, CP, to be let fall from the point C upon the side AB, produced if necessary; then, it is manifest that, the arcs AP and BP, or the angles AGP and BGP measured by these arcs, will be the segments formed by this perpendicular: it is likewise evident that, the half, G′C, of the chord G′Cg′ will be the *sine* of the said perpendicular, and consequently GC its *cosine*. This being premised, from the right-angled triangle GHC we shall first get, R : GC or *cos*. CP : : *sin*. AP : CH: also, since the lines, rG, dG, lH and CH, are proportional, we shall have, *cos*. A : R : : CH : *sin*. AC; consequently, by multiplying the corresponding terms of these two proportions together, we shall again have, *cos*. A : *cos*. CP : : *sin*. AP : *sin*. AC: we might find by a similar process that, *cos*. B : *cos*. CP : : *sin*. BP : *sin*. BC; and therefore that, *Sin*. AP : *sin*. BP : : *sin*. AC × *cos* A : *sin*. BC × *cos*. B : : *sin*. B × *cos*. A : *sin*. A × *cos*. B (by substituting *sin*. B : *sin*. A for *sin* AC : *sin*. BC) : : $\frac{sin.\,B}{cos.\,B} : \frac{sin.\,A}{cos.\,A}$: : *tang*. B : *tang*. A : : *cot*. A : *cot*. B; or, which is the same, that, *The sines of the segments of the base are to each other as the cotangents of their adjacent angles.* Q. E. 1°. I. & D.

2°. Since the angle GCH is the complement of the arc AP, and likewise the angle GCX the complement of the arc BP; the triangle GHC will give, R : *cos*.CP : : *cos*.AP : GH = *cos*.AC, and the triangle GCX, R : *cos*. CP : : *cos*. BP : GX = *cos*. BC : conse-

conſequently, ſince theſe proportions have their two firſt terms the ſame, *Coſ.* AP *will be* : *coſ.* BP : : *coſ.* AC : *coſ.* BC; or, *The coſines of the ſegments of the baſe as thoſe of the correſponding ſides.* Q. E. 2°. I. & D.

We might prove in the ſame manner, by letting fall the perpendicular Aδ upon the baſe BC, that, *Sin.* Bδ : *ſin.* Cδ : : *cot.* B : *cot.* C, and, *Coſ.* Bδ : *coſ.* Cδ : : *coſ.* AB : *coſ.* AC.

3°. Suppoſe now that the perpendicular Aδ is in reality let fall, and that the ſegments, BAδ, CAδ, of the angle BAC are actually taken into conſideration: then, as we have already proved in *art.* 206, that Rθ = Aδ, $\theta\zeta$ will of courſe be equal to the *coſine* of this perpendicular. This being granted, on account of the proportional lines, RG, ρG, $\theta\zeta$ and Zζ, and ſince ρG is manifeſtly equal to the *coſine* of the angle BAδ, we ſhall have, R : *coſ.* BAδ : : *coſ.* Aδ : ζZ : moreover, ſince ZG is (by *art.* 205) = *ſin.* B, and the angle ZGζ equal to the complement of AB, *coſ.* AB *will be* : R : : Zζ : *ſin.* B; therefore, if we multiply the correſponding terms of theſe two proportions together, we ſhall get, *coſ.* AB : *coſ.* BAδ, : : *coſ.* Aδ : *ſin.* B : we might find in like manner that, *coſ.* AC : *coſ.* CAδ : : *coſ.* Aδ : *ſin.* C : conſequently, we ſhall alſo have, *Coſ.* BAδ : *coſ.* CAδ : : *ſin.* B × *coſ.* AB : *ſin.* C × *coſ.* AC : : *ſin.* AC × *coſ.* AB : *ſin.* AB × *coſ.* AC : : $\frac{\textit{coſ.}\ AB}{\textit{ſin.}\ AB}$: $\frac{\textit{coſ.}\ AC}{\textit{ſin.}\ AC}$: : *cot.* AB : *cot.* AC : whence we infer that, *The coſines of the ſegments of the vertical angle are as the cotangents of their adjacent ſides.* Q. E. 3°. I. & D.

4°. From the proportions between the *ſines* of the ſides and thoſe of their oppoſite angles in the right-angled triangles, AδB, AδC, we ſhall get; *ſin.* BAδ

× *ſin.*

$\times$ *ſin.* Aδ = *ſin.* Bδ $\times$ *ſin.* B, and, *ſin.* CAδ $\times$ *ſin.* Aδ = *ſin.* Cδ $\times$ *ſin.* C; and conſequently, *Sin.* BAδ : *ſin.* CAδ : : *ſin.* Bδ $\times$ *ſin.* B : *ſin.* Cδ $\times$ *ſin.* C : : *coſ.* B : *coſ.* C (becauſe it hath been demonſtrated at the end of the ſecond part of this article that, *ſin.* Bδ : *ſin.* Cδ : : *cot.* B : *cot.* C) : whence it appears that, *The ſines of the ſegments of the vertical angle are to each other as the coſines of the angles above the baſe.* Q. E. 4°. I. & D.

SCHOLIUM.

230. We may now perceive how all the Theorems neceſſary to the reſolution of right or oblique-angled ſpherical triangles may be found, either algebraically or geometrically, by means of the graphical conſtructions explained in the foregoing Chapter. But a little obſervation will ſuffice to convince us, that there are other methods of diſcovering theſe truths beſides thoſe already ſpecified : thus, for inſtance, we might have obtained the laſt analogies from the interſection of the chords, $\nu\delta\omega\mu$, $\eta\delta\varepsilon\varphi'$; or (if we conſider that the arcs, $p\gamma$, $p\beta$, are the meaſures of the ſegments, ACP, BCP, formed by the perpendicular, CP, with the ſides containing the angle C), from the interſection of the lines, $\gamma\Lambda$, $l'\lambda$, $\beta\Phi$, $f\varphi$, &c.

If we ſuppoſe the ſpherical triangle BAC to become a right-lined triangle, it will be eaſy to find how the four preceding analogies will be then expreſſed.——The firſt, *ſin.* Bδ : *ſin.* Cδ : : *cot.* B : *cot.* C, becomes, Bδ : Cδ : : *cot.* B : *cot.* C; and therefore in any plain triangle, *The ſegments of the baſe formed by a perpendicular let fall from the vertical angle are as the cotangents of their adjacent angles:* for, if Aδ be regarded as radius, theſe lines will be the *tangents* of the ſegments of the angle BAC, which are the complements of the angles above the baſe.

The

The ſecond analogy becomes, $\infty : \infty :: \infty : \infty$.
The third gives, *The coſines of the ſegments of the vertical angle in the ſame proportion to each other as are their oppoſite ſides*; becauſe the three angles of any plain triangle, when taken together, are only equal to two right angles. Laſtly, the fourth analogy gives a proportion whereof the alternate terms are identically the ſame, viz. the antecedent equal to the antecedent, and the conſequent to the conſequent.

COROLLARY.

231. If the arc Aδ be ſuppoſed to divide the angle BAC into two equal parts; then, ſince the angles, AδB, AδC, which are the ſupplements to each other, have the ſame *ſine*, and, *ſin.* AδB : *ſin.* AB : : *ſin.* BAδ : *ſin.* Bδ, as alſo, *ſin.* AδC : *ſin.* AC : : *ſin.* CAδ : *ſin.* Cδ; it will follow that in this caſe, *The ſines of the ſegments of the baſe are as the ſines of the ſides oppoſite to them*; and conſequently, in a right-lined triangle, *Theſe ſegments will be to each other as their oppoſite ſides*, whenever the angle included between theſe ſides is biſected.

SCHOLIUM.

232. If we look back upon the ſeveral formulæ which we have diſcovered by the algebraical analyſis, we ſhall perceive that, in order to find the logarithm of any unknown quantity, it is generally requiſite to find that of the ſum or difference of two given quantities. Now we have already ſhewn the method of performing theſe operations, and therefore it will be ſufficient in this place to make application thereof to a few examples.

EXAMPLE I.

Fig. 11. 233. Suppoſe that in the triangle BAC we know the two ſides, AB, BC and the included angle B to be, 41° 9′, 71° 30′ and 27° 8′ reſpectively; it is required

required to find the angle at A by the formula in *art.* 214, $tang.\ A = \frac{sin.B \times RR}{sin.AB \times cot.BC - cos.AB \times cos.B}$; where we assume the sign —, because the angle at A is to be acute. Now, it is manifest that, the whole difficulty in performing the operation will consist in finding the logarithm of the denominator; but this will be easily surmounted, by regarding *sin.* AB × *cot.* BC as one angle, and *cos.* AB × *cos.* B as another, and afterwards finding the *sine* of their difference by the formula in *art.* 92, *sin.* A — *sin.* B $= 2\,cos.\ \overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times sin.\ \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.

OPERATION *by the* LOGARITHMS.

9.818247 = *log. sin.* AB.	9.948331 = *log. cos.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$.
9.524520 = *log. cot.* BC.	9.403790 = *log. sin.* $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.
9.342767 = *log. sin.* 12° 43′ 20″ = B.	0.301030 = *log.* 2.
9.876789 = *log. cos.* AB.	19.653151 = *log. to be subtr. from*
9.949364 = *log. cos.* B.	29.659025 = *log. sin.* B × RR.
9.826153 = *log. sin.* 42° 4′ 35″ = A.	10.005874 = *log. tang.* 45° 23′ 14″.

EXAMPLE II.

234. Suppose the same things given as in the preceding Example; it is required to find the third side, AC, by the formula in *art.* 216, $cos.\ AC = \frac{cos.B \times sin.AB \times sin.BC + cos.AB \times cos.BC \times R}{RR}$. —— If this quantity be considered as the expression of a *sine* equal to the sum of the *sines* of two angles, and the formula in *art.* 91, *sin.* A + *sin.* B $= 2\,sin.\ \overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times cos.\ \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$, made use of; the operation will be perfectly easy, and performed as follows:

9.949364 = *log. cos.* B.	$\tfrac{1}{2}A + \tfrac{1}{2}B$ = 23° 46′ 44″.
9.818247 = *log. sin.* AB.	*comp.* $\tfrac{1}{2}A - \tfrac{1}{2}B$ = 80° 2′ 39″.
9.976957 = *log. sin.* BC.	
9.744568 = *log. sin.* 33° 44′ 5″ = A.	9.605529 = *log. sin.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$.
9.876789 = *log. cos.* AB.	9.993410 = *log. cos.* $\tfrac{1}{2}A - \tfrac{1}{2}B$.
9.501476 = *log. cos.* BC.	0.301030 = *log. of* 2.
9.378265 = *log. sin.* 13° 49′ 23″ = B.	9.899969 = *log. cos.* 37° 24′ 54″.

Example III.

235. Suppose now that the three sides, AB, AC and BC, of the triangle BAC are given, viz. AB $= 41° 9'$; AC $= 52° 35'$, and BC $= 71° 30'$; it is required to find one of the angles of this triangle, as the angle A for instance, by the formula in *art.* 225, $cof. A = \frac{cof.BC \times RR - cof.AC \times cof.AB \times R}{fin.AB \times fin.AC}$. —— The value of this expression will be obtained by means of the formula in *art.* 92, *fin.* A — *fin.* B, *&c.* and the operation stand thus:

$$
\begin{array}{ll}
9.783623 = & log.\ cof.\ AC. \\
\underline{0.876789} = & log.\ cof.\ AB. \\
9.660412 = & log.\ fin.\ 27° 13' 38'' = A. \\
 & \underline{cof.BC = 18° 30' \qquad = B.} \\
 & therefore,\ 22° 51' 49'' = \tfrac{1}{2}A + \tfrac{1}{2}B. \\
 & and,\ 4° 21' 49'' = \tfrac{1}{2}A - \tfrac{1}{2}B. \\
0.181753 = & arith.\ comp.\ fin.\ AB. \\
0.100049 = & arith.\ comp.\ fin.\ AC. \\
0.301030 = & log.\ of\ 2. \\
9.964463 = & log.\ cof.\ \overline{\tfrac{1}{2}A + \tfrac{1}{2}B}. \\
\underline{8.881303} = & log.\ fin.\ \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}. \\
\underline{9.428598} = & log.\ cof.\ 74° 26' 15'' = cof.\ A.
\end{array}
$$

236. Since, in the solutions of the sixth and seventh Problems, in *art.* 223 and 224, equations of the second degree have been concerned, it is manifest that, there may be certain cases wherein we may be obliged to solve equations of this nature; and as, moreover, all the calculations of practical Astronomy are made by the logarithms, it will not be improper to shew here the manner of obtaining these solutions by the Tables of *fines*, *&c.* for which reason we shall annex the two following Problems.

Problem I.

237. *To find by the Tables of fines, &c. the roots of the equation,* $x^2 \pm 2ax = bb$.

SOLU-

SOLUTION.

From any point, C, with a radius equal to a, let the circle ADBd be described, and at the extremity of the radius CA the perpendicular AL erected, equal to the quantity b: then, through the centre C and point L, let the line LDCd be drawn, terminated by the circumference at d; and the parts, LD, Ld, will be the roots of the equation. The demonstration of this is very obvious from the application of Algebra to Geometry; for the *tangent* AL and *secant* Ld give, Ld : LA : : LA : LD, or, algebraically, $a + \sqrt{aa + bb} : b :: b :$ LD (x); whence we deduce, $ax + x\sqrt{aa+bb} = bb$, or, $x\sqrt{aa+bb} = bb - ax$; and consequently, by squaring both sides of the equation, striking out the terms that destroy, and dividing the remainder by bb, $x^2 + 2ax = bb$. Fig. 24.

Now, in order to reduce this construction to the logarithms of the Tables, let there be again erected the right line FLf perpendicular to the extremity of AL, and terminated at, F, f, by the prolongations of the chords, AD, Ad; and let the chords, BD, Bd, be drawn. —— This being perfectly understood, it is easy to perceive that, in the right-angled triangle CAL, the angle ACL will, in the first place, be determined by this analogy; CA (a) : AL (b) : : r : *tang*. ACL: then, on account of the similar triangles, BDA, ALF, we shall have, BD : DA : : AL : LF* = LD (x): but the angle ABD is evidently the half of the angle ACL before found, and therefore we shall lastly have, R :

 tang.

* The triangles ADB and ADd are manifestly equal to each other, and also similar to the triangle ALF; whence, the angle ADd = the angle FDL = the angle AFL, and consequently, LF = LD.

tang. $\frac{1}{2}$ ACL : : AL (b) : LF or LD (x). Consequently, from these proportions the value of x will be obtained by the logarithms. Q. E. 1°. I.

2°. If Ld be called x, we shall find from the same proportion (Ld : LA : : LA : LD) that, $x^2 - 2ax = bb$. Now, to obtain the value of x in this case, we must first say, as before, $a : b :: r$: *tang.* ACL : then, on account of the similar right-angled triangles, BdA, ALf, we shall have, Bd : dA : : AL : Lf = Ld; but, Bd : dA : : R : *cot. half the angle* BCd = ACL; therefore, we shall lastly have, r : *cot. of half the angle found by the last proportion* : : *the quantity* b : *to the line* Lf; which will be the positive root of the equation, $xx - 2ax = bb$. Q. E. 2°. I.

PROBLEM II.

238. *To find by the Tables of sines, &c. the roots of the equation,* $x^2 \pm 2ax + bb = 0$.

SOLUTION.

Fig. 25. At the extremity of the radius CA = a let there be erected a perpendicular AL = b; then through L let there be drawn the line LFf parallel to AB (which will cut the circle described upon AB in two points F and f, or otherwise touch it in only one point, if the Problem be possible), and, LF, Lf, will be roots of the equation, $x^2 \pm 2ax + bb = 0$. For, calling FL, x; on account of the *tangent* AL and *secant* Lf, we shall have, Lf ($a + \sqrt{aa - bb}$) : AL (b) : : AL (b) : LF (x); from whence we shall easily get, by taking the product of the extremes and means, $x^2 - 2ax + bb = 0$. We might have found in like manner, by calling Lf, x; $x^2 - 2ax + bb = 0$. This being premised, if we again draw the lines, AF, Af, FB, FC, FD and fd, we shall, in the first place, get from the right-angled triangle

triangle CDF, CF : FD or $a : b :: r$: *ſine of the angle* FCD : then, on account of the ſimilar triangles, BFA, FDA, we ſhall have, BF : FA :: FD : DA; but the angle at B is manifeſtly the half of the angle ACF at the centre, and therefore we ſhall have, BF : FA :: R : *tang.* $\frac{1}{2}$ FCA : conſequently, we ſhall laſtly have, R : *tang.* $\frac{1}{2}$ FCA :: FD (b) : AD (x); from whence it follows that AD or x will become known, ſince the three firſt terms of this proportion are known.

2°. If Lf be called x, we ſhall in this caſe have, on account of the ſimilar triangles, AFB, fdA; AF : BF :: fd : dA$=$Lf; or, r : *cot.* $\frac{1}{2}$ FCA :: $b : x$*. Q. E. I.

We ſometimes find in Aſtronomy ſuch Problems as lead us into equations of the third degree, but particularly in the calculations of the motions of Comets in a parabolical orbit; for which reaſon we hope our readers will not be diſpleaſed to ſee here the method of ſolving equations of *this* kind likewiſe by the Tables of *ſines*, &c. But before we do this, we ſhall premiſe in a few words what relates to theſe equations in general; and refer, for the demonſtration of the different articles, to what Mr. *Clairaut* hath written upon this ſubject in his Elements of Algebra.

We will ſuppoſe then as truths demonſtrated in ſeveral Treatiſes of Algebra:

1°. That every equation of the third degree may be reduced to the form, $x^3 \pm px \pm q = 0$; wherein the ſecond term is taken away.

2°. That of the roots of the equation, $x^3 + px \pm q = 0$, there are neceſſarily two imaginary and one real.

3°. That

* For theſe two Problems ſee *Simpſon*'s Trigonometry, p. 64, &c. 2d edition.

3°. That in the equation, $x^3 - px \pm q = 0$, there are neceſſarily two imaginary roots, whenever $\frac{1}{27}p^3$ is leſs than $\frac{1}{4}qq$.

4°. That, on the contrary, the ſame equation will have its three roots real, when $\frac{1}{27}p^3$ (always negative) is greater than $\frac{1}{4}qq$; which conſtitutes the caſe called the *irreducible* caſe.

Now though theſe conſiderations are not abſolutely neceſſary to the underſtanding of the ſubſequent ſolutions, which are in a great meaſure independent thereon; yet they will ſerve to convince us that theſe ſolutions contain all the poſſible caſes of cubic equations, after the ſecond term is taken away, according to the common and well known methods of transformations.

PROBLEM I.

240. *To find the three roots of an equation of the third degree of the form,* $x^3 - px \pm q = 0$, *by ſuppoſing* $\frac{1}{27}p^3$ *greater than* $\frac{1}{4}qq$; that is to ſay, *in the irreducible caſe.*

SOLUTION.

Let the equation, $4x^3 - 3r^2x \pm r^2y = 0$, be aſſumed (which by means of the indeterminate quantity y may repreſent all cubic equations belonging to the irreducible caſe), and let r be ſuppoſed therein to be greater than y, in order to have $\frac{1}{27}p^3$ greater than $\frac{1}{4}qq$: then, if the laſt term hath the ſine + prefixed to it, put $y = ſin.\ A$, and the circumference of a circle $= C$; and the roots of the equation will be, $x = ſin.\ \frac{A}{3}$; $x = ſin.\ \frac{A+C}{3}$, and, $x = ſin.\ \frac{A+2C}{3}$. ——If the laſt term hath the ſign — prefixed to it, let y be put $= coſ.\ A$, and the roots of the equation ($4x^3 - 3r^2x - r^2y = 0$) will be, $x = coſ.\ \frac{A}{3}$; $x =$

coſ.

$cos. \frac{A+C}{3}$, and, $x = cos. \frac{A+2C}{3}$. —— The demonstration of these operations is very easily deduced from the general formulæ in *art.* 53 & 54, concerning the *sines* and *cosines* of multiple arcs. Q.E.I.

PROBLEM II.

241. *To find the real root of the equation,* $x^3 - px \mp q = 0$, *when* $\frac{1}{27}p^3$ *is supposed to be less than* $\frac{1}{4}qq$; *the other two roots being, in that case, imaginary.*

SOLUTION.

Let the same equation ($4x^3 - 3r^2x \mp r^2y = 0$) be assumed, as before; in which let r be supposed less than y, that $\frac{1}{27}p^3$ may be less than $\frac{1}{4}qq$: then, if the last term be supposed negative, put $y = cosec.\, 2A$ (which will necessarily become known when y is determined); after which find an angle, B, whose $cot.^3$ may $= r^2\, cot.$ A, and the value of x will be $cosec.$ 2B. —— If the last term be affirmative, we must then consider y as the *cosecant* of an obtuse angle, and the solution of the Problem will be obtained in the same manner. Q. E. I.

PROBLEM III.

242. *To find the real root of the equation,* $x^3 + px \mp q = 0$; *the other two roots being imaginary, since* $\frac{1}{27}p^3$ *will necessarily be positive.*

SOLUTION.

Let the same equation ($4x^3 + 3r^2x \mp r^2y = 0$) be still retained; in which make $y = cot.\, 2A$, and afterwards find an angle, B, so that $cot.^3$ B may $= r^2\, cot.$ A; and the value of the unknown quantity x will be $= cot.$ 2B. Q. E. I.

DEMONSTRATION *of the two last Solutions.*

243. Whether r be greater or less than y, it may always be truly affirmed that the equation, $4x^3 -$

$3r^2x$

$3r^2x - r^2y = 0$, is the ſame with, $\overline{x + \sqrt{x^2 - r^2}}|^3$ $= r^2 \times \overline{y + \sqrt{y^2 - r^2}}$; and conſequently that the value of x in y obtained from the latter, will ſolve the former equation. All that is requiſite to be done then is to prove the identity of theſe two expreſſions. And in order to this, we ſhall in the firſt place obſerve, that by the formula in *art.* 29 we in reality have, $4x^3 - 3r^2x - r^2y = 0$: ſo much granted, let it, in the next place, be allowed as a ſuppoſition that, $\overline{x + \sqrt{x^2 - r^2}}|^3 = r^2 \times \overline{y + \sqrt{y^2 - r^2}}$; then by expanding the cube we ſhall get, $4x^3 - 3r^2x - r^2y + \overline{4x^2 - r^2} \times \sqrt{x^2 - r^2} = r^2\sqrt{y^2 - r^2}$; but herein the terms, $4x^3 - 3r^2x - r^2y$, occur, which have been already put $= 0$, and therefore it remains only to be proved that, $\overline{4x^2 - r^2} \times \sqrt{x^2 - r^2}$ $= r^2\sqrt{y^2 - r^2}$. Now let both ſides of this equation be raiſed to the ſquare, and we ſhall find, $16x^6 - 24r^2x^4 + 9r^4x^2 - r^6 = r^4y^2 - r^6$, or, $16x^6 - 24r^2x^4 + 9r^4x^2 = r^4y^2$; which is exactly the ſquare of the equation, $4x^3 - 3r^2x = r^2y$; and therefore the truth of what was required to be proved follows of courſe.

If now as in *Prob.* II. we ſuppoſe r to be leſs than y, and put $y = cosec.\ 2A$, we ſhall get, $\sqrt{y^2 - r^2}$ $= cot.\ 2A$: likewiſe, if $x = cosec.\ 2B$, $\sqrt{x^2 - r^2}$ will be $= cot.\ 2B$: therefore, by ſubſtituting theſe values in the equation $\overline{x + \sqrt{x^2 - r^2}}|^3 = r^2 \times \overline{y + \sqrt{y^2 - r^2}}$, we ſhall get, $\overline{cosec. 2B + cot. 2B}|^3 = r^2 \times \overline{cosec. 2A + cot. 2A}$; and ſince, by the formula in *art.* 84, we have, $cosec.\ 2A + cot.\ 2A = cot.\ A$, the laſt equation, after proper ſubſtitution in each member thereof, will become, $cot.^3\ B = r^2\ cot.\ A$: from whence it will be eaſy to find $cot.\ B$, when A is known; and conſequently the value of $x =$ by ſuppoſition to $cosec.\ 2B$. Q. E. 1°. D.

2°. We

2°. We may prove in the same manner that the equation, $4x^3 + 3r^2x - r^2y = 0$, is identically the same with, $\overline{x + \sqrt{x^2 + r^2}}|^3 = r^2 \times \overline{y + \sqrt{y^2 + r^2}}$: but this equation, by supposing as in *Prob.* III. $y = cot. 2A$, and $x = cot. 2B$, will become, $\overline{cot. 2B + cosec. 2B}|^3 = r^2 \times \overline{cot. 2A + cosec. 2A}$; and in the last place, $cot.^3 B = r^2 \, cot. A$: whence it will be easy to find the arc denoted by B, when the value of A is known; and consequently *cot.* 2B, which is the value of the unknown quantity x in the equation. Q. E. 2°. D.

We shall now, in order to make the utility and nature of these solutions more clearly appear, apply them to particular examples.

EXAMPLE I.

244. *To find the roots of the equation,* $x^3 - 3x - 1 = 0$; *which will be all three real, because* $\frac{1}{27}p^3$ *is negative, and greater than* $\frac{1}{4}qq$.

SOLUTION.

That this equation may be compared with $4x^3 - 3r^2x - r^2y = 0$, I first divide all the terms of the latter by 4, and find, $x^3 - \frac{3}{4}r^2x - \frac{1}{4}r^2y = 0$: then will $\frac{3}{4}r^2 = 3$; whence $r = 2$, and consequently $y = 1$. —— Now because the last term is negative, y, according to what has been said in *Prob.* I. *art.* 240, must be put $=$ *cos.* A; and since this *cosine* is half the sine total, it must necessarily be the *sine* of an angle of 30°, and therefore A itself $= 60°$. Consequently, the values of x will be, *cos.* 20°; *cos.* 140°, and, *cos.* 260°; or, *sin.* 70°; —*sin.* 50°, and, —*sin.* 10°, to a radius expressed by 2; that is to say, the real values of x will be double the *sines* corresponding to these angles in the Tables.

EXAMPLE II.

245. *To find the real root of the equation,* $x^3 - x - 6 = 0$; *the other two being necessarily imaginary, since* $\frac{1}{27}p^3$ *is less than* $\frac{1}{4}qq$.

SOLUTION.

This equation I compare with, $4x^3 - 3r^2x - r^2y = 0$, after dividing all the terms of the latter by 4, and find $\frac{3}{4}r^2 = 1$; whence $r = \frac{2}{\sqrt{3}}$, and $y = 18$, which must be put $=$ *cosec.* 2A. Then, to obtain this angle by the Tables, I in the first place say, $\frac{2}{\sqrt{3}} : 18 :: $ *sine total of the Tables* $(1) : 9\sqrt{3} =$ *cosec.* 2A: but, by *art.* 81, *cosec.* A $= \frac{rr}{\textit{sin.}\,A}$, and therefore, *sin.* 2A $= \frac{rr}{\textit{cosec.}\,2A}$: whence, by subtracting from double the logarithm of the sine total, the logarithm of $9\sqrt{3}$, *sin.* 2A will be found $= 0.807196$, or, according to the index in the Tables, $8.807196 = 3^\circ\ 40'\ 40''$; and consequently, $A = 1^\circ\ 50'\ 20''$.——In the next place, to find the angle B, which ought to determine the value of x by the *cosecant* of its double; to the number 11.493417 let twice the logarithm of the sine total be added, and a third of the resulting sum taken, and the quotient will give, *cot.* B $=$ *cot.* $17^\circ\ 37'\ 50''$, or, $B = 17^\circ\ 37'\ 50'$; the *cosecant* of the double whereof, that is, the *secant* of $54^\circ\ 44'\ 20''$, multiplied by $\frac{2}{\sqrt{3}}$ will be the root required of the proposed equation.

EXAMPLE III.

246. *To find the real root of the equation,* $25x^3 + 75x - 46x = 0$; *the other two being necessarily imaginary, because the second term is affirmative.*

SOLUTION.

That this equation may be compared with, $4x^3 + 3r^2x - r^2y = 0$, I first divide all the terms of the

the latter by 4, and those of the proposed one by 25; after which I deduce, from a comparison of the corresponding terms of the two resulting equations ($x^3 + \frac{3}{4}r^2x - \frac{1}{4}r^2y = 0$, and, $x^3 + 3x - \frac{46}{25} = 0$), $r = 2$, and $y = \frac{46}{25}$. This done, in order to find what the angle A must be, so that $\frac{46}{25}$ may be the *cotangent* of its double, I make this proportion; *radius (before found* 2) : $\frac{46}{25}$: : *sine total of the Tables* (1.000000) : *cot.* 2A *in Prob.* III. but, by *art.* 12, *tang.* $A = \frac{rr}{cot.\ A}$, or, *tang.* $2A = \frac{rr}{cot.\ 2A}$; and therefore, by performing the necessary logarithmical operation, *tang.* 2A will be found = 10.036212 = 47° 23′ 10″, and consequently, A = 23° 41′ 35″. The value of A being thus obtained, that of B may be found as in the preceding Problem to be = 37° 13′ 54″; and if the *cotangent* of twice this angle be multiplied by 2 (because $r = 2$, as above said), the product will be the root of the equation. Q. E. I.

OBSERVATION.

It may be observed, that these solutions of equations of the third degree by the circle necessarily contain those of equations of the fourth degree likewise; since it is demonstrated in most Treatises of Algebra, that equations of this degree may be reduced to those of the third. It may likewise be observed, that it would be very easy to find particular equations in any degree which, as well as the foregoing ones, would admit of solutions by means of the circle; but we think it unnecessary to specify instances, as what we have already said will be sufficient for common practice.

CHAP. V.

Containing the fluxional Analogies of spherical and plain Triangles.

SECT. I.

LEMMA I.

247. *The fluxion of an arc is to that of its sine, as radius is to the cosine of this arc.*

DEMONSTRATION.

Fig. 26. LET there be an arc, AM, which we shall call z; of which let the *sine*, MP, be denoted by s; the *cosine*, CP, by u; the *tangent*, AT, by t, and the *secant*, CT, by y. Then we must prove that, $\dot{z} : \dot{s} :: r : u$.

Now, in order to this, let us suppose the arc AM to become Am, and let the lines, Cmt, mp, be drawn; also, from the centre C with the radius CT, let the arc TS be described, and through the point M the line Mr drawn parallel to CA: then it is manifest by this construction that, Mm will be $= \dot{z}$; $mr = \dot{s}$; M$r = -\dot{u}$ (because the arc AM increasing, its *cosine* decreases; T$t = \dot{t}$, and, tS $= \dot{y}$. So much being premised, on account of the similar triangles, mrM, CPM, we shall have, CM (r) : CP (u) :: Mm $(\dot{z})$: mr $(\dot{s})$. Q. E. D.

LEMMA II.

248. *The fluxion of an arc is to that of its tangent, as radius squared is to the square of the secant*; that is, $\dot{z} : \dot{t} :: rr : yy$.

DEMONSTRATION.

Fig. 26. On account of the similar sectors, CMm, CTS, we shall have, CM (r) : CT (y) :: Mm $(\dot{z})$: TS $=$

$y\dot{z}$.

$\frac{y\dot{z}}{r}$: likewise, on account of the right-angled triangles, CAT, TSt, which are also similar (since, when the lines, CT, Ct, are indefinitely near to a coincidence, the angle at t will not essentially differ from that at T), we shall have, CA : CT :: ST : Tt; or, $r : y :: \frac{y\dot{z}}{r} : \dot{t}$; whence we immediately deduce, $\dot{z} : \dot{t} :: r^2 : y^2$. Q. E. D.

Lemma III.

249. *The fluxion of an arc is to that of its secant, as radius squared is to the rectangle under the secant and tangent*; that is, $\dot{z} : \dot{y} :: r^2 : ty$.

DEMONSTRATION.

It is evident that St is the fluxion of the *secant*, and ST hath already been found to be $= \frac{y\dot{z}}{r}$; therefore, by again comparing the homologous sides of the similar triangles, CAT, TSt, we shall get, CA : AT :: ST : St; or, $r : t :: \frac{y\dot{z}}{r} : \dot{y}$; and consequently, $\dot{z} : \dot{y} : r^2 :: ty$. Q. E. D.

COROLLARY.

250. Hence, by collecting these different expressions, we shall have for the fluxion of any arc z; $\dot{z} = \frac{r\dot{s}}{u} = -\frac{r\dot{u}}{s}$ (by making use of the fluxion of the *cosine*) $= \frac{rr\dot{t}}{yy} =$ (by *art.* 15) $\frac{u\dot{t}}{y} = \frac{r^2\dot{y}}{ty} =$ (by *art.* 11) $\frac{ru\dot{y}}{sy}$. —— We might proceed to find with equal facility other formulæ for the *cotangents* and *cosecants*; but those which we have already found will be sufficient for all that we shall have to demonstrate in this Chapter.

SCHO-

SCHOLIUM.

251. We might have demonſtrated the ſeveral preceding analogies by the formulæ obtained in the firſt Chapter. For inſtance, the firſt Lemma might have been immediately deduced from *art.* 85, $\mathit{ſin.}\ \overline{A + B} = \mathit{ſin.}\ A \times \mathit{coſ.}\ B + \mathit{coſ.}\ A \times \mathit{ſin.}\ B$: for, $\dot{\mathit{ſin.}}\ AM = \mathit{ſin.}\ \overline{\dot{A}M + mM} - \mathit{ſin.}\ AM$; but, in an evaneſcent arc, $\dot{z}$, the *coſine* is equal toradi us or unity, and the *ſine* coincides with the arc itſelf; wherefore we ſhall have, $\mathit{ſin.}\ \overline{z + \dot{z}} = \mathit{ſin.}\ z + \frac{\dot{z}\ \mathit{coſ.}\ z}{r} = s + \frac{u\dot{z}}{r}$, and conſequently, $\dot{s}$ or $\mathit{ſin.}\ \overline{z + \dot{z}} - s = s + \frac{u\dot{z}}{r} - s = \frac{u\dot{z}}{r}$. —— The other formulæ might have been as eaſily deduced from the formulæ inveſtigated in the ſame Chapter.

Lemma IV.

252. *If from the three angles of any ſpherical tri-*
Fig. 11. *angle*, BAC, *as poles, there be deſcribed upon the ſurface of the ſphere another ſpherical triangle*, DEF; *the fluxions of the angles*, D, E, F, *of this triangle will be reſpectively equal to thoſe of the oppoſite ſides*, AB, AC, BC, *of the triangle* BAC, *and the fluxions of the ſides*, DE, EF *and* FD, *of the ſaid triangle the ſame as thoſe of the oppoſite angles in the triangle* BAC; *and contrarily, for the fluxions of the parts of the triangle* BAC *with reſpect to the parts of the triangle* DEF.

DEMONSTRATION.

This propoſition is an evident conſequence of what hath been demonſtrated in *art.* 130, and of this; that angles, which are the ſupplements to each other, have neceſſarily the ſame *ſine*, *coſine*, *tangent*, *cotangent*, *ſecant* and *coſecant*. Q. E. D.

SECT.

SECT. II.

Concerning the fluxions of any ſpherical or plain triangle, wherein an angle and one of its adjacent ſides are ſuppoſed conſtant.

THEOREM I.

253. *If in any ſpherical triangle,* BAC, *we ſuppoſe an angle,* A, *conſtant, together with one of the ſides,* AC, *adjacent to this angle, we ſhall always have this analogy; as the fluxion of the other ſide adjacent to the conſtant angle is to that of the ſide oppoſite, ſo is radius to the coſine of the angle oppoſite to the conſtant ſide*; Fig. 27.
that is, $\dot{\overline{AB}} : \dot{\overline{BC}} :: R :$ *coſ.* B.

DEMONSTRATION.

Suppoſing the ſide AB to flow into Aβ, or to be increaſed by the indefinitely ſmall quantity Bβ; if Cβ be drawn, and C*b* taken thereon = CB, by deſcribing from the pole C the little circular arc B*b*; *b*β will be the fluxion of the ſide CB, and the little triangle B*b*β will be right-angled at *b*, as alſo (becauſe on account of the ſmallneſs of its ſides it may be conſidered as right-lined) have its ſides proportional to the *ſines* of their oppoſite angles: but the angle at β is eſſentially equal to the angle B, and conſequently the angle βB*b* the complement of the ſame angle; wherefore we ſhall have, Bβ : *b*β : R : : *coſ.*B, or, $\dot{\overline{AB}} : \dot{\overline{BC}} :: R :$ *coſ.* B. Q. E. D.

COROLLARY I.

254. From this proportion, and the formulæ demonſtrated in the preceding Chapter, the following analogies may be eaſily deduced; $\dot{\overline{AB}} : \dot{\overline{BC}} ::$ *ſin.* AB × *ſin.* BC : R × *coſ.* AC — *coſ.* AB × *coſ.* BC (by ſubſtituting for *coſ.* B its value in *art.* 225) : :

$$R : \frac{cos.\,AC \times sin.\,A \times sin.\,C}{RR} - \frac{cos.\,A \times cos.\,C}{R} \text{ (by } art.\,221\text{)}$$

: : RR

: : RR : *cos*. AC × *sin*. C (if the angle A be 90°) : : *tang*. BC : *tang*. AB, by *art*. 228.

Corollary II.

255. If the triangle is right-lined, we shall still have under the same suppositions, $\dot{\overline{AB}} : \dot{\overline{BC}}$: : R : *cos*. B.

Theorem II.

256. *If the same things be supposed as in the preceding Theorem; the fluxion of the variable side adjacent to the constant angle is to that of the angle opposite to this side, as the sine of the side opposite to the constant angle is to the sine of the angle opposite to the constant side*; that is, $\dot{\overline{AB}} : \dot{C}$: : *sin*. BC : *sin*. B.

DEMONSTRATION.

Let the sides, CB, C*b*, be produced till they be each 90°; and, it is manifest that, the arc F*f* will be the measure of the variation of the angle C. This being granted, since the little arcs, F*f*, B*b*, contain the same number of degrees, we shall have, F*f* : B*b* : : R : *sin*. BC; and likewise, on account of the triangle B*b*β, B*b* : Bβ : : *sin*. B : R: then, if we multiply the corresponding terms of these two proportions together, we shall get, F*f* : Bβ : : *sin*. B : *sin*. BC; or, $\dot{C} : \dot{\overline{AB}}$: : *sin*. B : *sin*. BC, and, *invertendo*, $\dot{\overline{AB}} : \dot{C}$: : *sin*. BC : *sin*. B. Q.E.D.

Corollary I.

257. If for the quantities, *sin*. B and *sin*. BC, there be substituted their different values deduced from the proportion between the *sines* of angles and those of their opposite sides, we shall find this series of equal comparisons; $\dot{\overline{AB}} : \dot{C}$: : $sin.^2$ BC : *sin*. AC × *sin*. A : : *sin*. AC × *sin*. A : $sin.^2$ B : : *sin*. BC × *sin*. AB : *sin*. C × *sin*. AC : : *sin*. A × $sin.^2$ AB : *sin*. AC × $sin.^2$ C.

COROLLARY II.

258. If the triangle be right-lined, the *ſine* of the ſide BC will then become the ſide itſelf, and therefore the Theorem expreſſed thus; $\overline{\dot{AB}} : \dot{C} :: BC : fin. B$.

THEOREM III.

259. *The fluxion of the ſide oppoſite to the conſtant angle is to that of the variable angle adjacent to the conſtant ſide, as the ſine of this ſide multiplied by the coſine of the third angle is to the ſine of the ſame angle multiplied by the radius*; that is, $\overline{\dot{BC}} : \dot{C} :: fin. BC \times cof. B : fin. B \times R :: fin. BC : tang. B$.

DEMONSTRATION.

Since we have, by *art.* 253, $\overline{\dot{BC}} : \overline{\dot{AB}} :: cof. B : R$; and alſo, by the laſt Theorem, $\overline{\dot{AB}} : \dot{C} :: fin. BC : fin. B$; therefore we ſhall get, by multiplying the correſponding terms of theſe two proportions together, $\overline{\dot{BC}} : \dot{C} :: fin. BC \times cof. B : fin. B \times R :: fin. BC : tang. B$. Q. E. D.

COROLLARY I.

260. If we ſubſtitute for *ſin.* BC its value, $\frac{fin. AC \times fin. A}{fin. B}$, this new analogy will ariſe, $\overline{\dot{BC}} : \dot{C} :: fin. A \times fin. AC : tang. B \times fin. B :: fin.^2 BC \times cot. AC - fin. BC \times cof. BC \times cof. C : R^2 \times fin. C$, by *art.* 214.

COROLLARY II.

261. If the triangle be right-lined, the ſame ſuppoſitions will give, $\overline{\dot{BC}} : \dot{C} :: BC : tang. B$; for the *ſine* of BC then becomes the ſide BC itſelf.

THEOREM IV.

262. *Let the ſame things be ſtill ſuppoſed, and I ſay that; the fluxion of the ſide adjacent to the conſtant angle*

angle is to that of the angle adjacent to this side, as the tangent of the side opposite to the constant angle is to the sine of the angle opposite to the constant side; that is, $\dot{\overline{AB}} : \dot{B} :: tang.\, BC : sin.\, B$.

DEMONSTRATION.

According to the suppositions of the Theorem,
Fig. 11. in the triangle DEF, whose parts are all, by construction, supplements to those of the triangle BAC, the angle at E and the side DE, adjacent thereto, will be constant. Moreover, by the fourth Lemma of this Chapter, the fluxions of the angle D and side DF will be respectively equal to those of the side AB and angle B of the triangle BAC: but, by the last Theorem, we have, $\dot{\overline{DF}} : \dot{D} :: sin.\, DF : tang.\, F$; and therefore, since angles supplemental to each other have the same *sine* and *tangent*, we shall get, by substituting for these quantities their corresponding values in the triangle BAC, and, *invertendo*, $\dot{\overline{AB}} : \dot{B} :: tang.\, BC : sin.\, B$. Q. E. D.

COROLLARY I.

263. Since we have, $\dot{\overline{AB}} : \dot{B} :: tang.\, BC : sin.\, B$, and also, by *art.* 256, $\dot{C} : \dot{\overline{AB}} :: sin.\, B : sin.\, BC$; we shall find, by multiplying these two proportions together, $\dot{B} : \dot{C} :: sin.\, BC : tang.\, BC :: cos.\, BC : R$ (*art.* 11) $:: cos.\, A \times R - cos.\, B \times cos.\, C :: sin.\, B \times sin.\, C$ (*art.* 226) $:: cos.\, A \times sin.\, AC \times sin.\, AB + cos.\, AC \times sin.\, AB \times R : R^3$, by putting for *cos.* BC its value in *art.* 216.

COROLLARY II.

264. If the triangle be right-lined, *sin.* BC will then be = *tang.* BC, and therefore the fluxions of the angles B and C also equal: which thing must of necessity happen; since in every right-lined triangle

angle the ſum of the three angles is always a conſtant quantity, and moreover by ſuppoſition one of the angles, A, conſtant.

SECT. III.

Concerning the fluxions of any ſpherical or right-lined triangle, wherein one of the angles and its oppoſite ſide are ſuppoſed conſtant.

THEOREM V.

265. *Suppoſe that in the triangle* BAC *the angle at* A *and its oppoſite ſide*, BC, *are conſtant; the fluxion of either of the other angles will be to that of its oppoſite ſide, as the tangent of this angle is to the tangent of the ſame ſide*; that is, $\dot{B} : \dot{\overline{AC}} :: tang.\,B : tang.\,AC$; or, $\dot{C} : \dot{\overline{AB}} :: tang.\,C : tang.\,AB$. Fig. 28.

DEMONSTRATION.

Since the angle at A and the oppoſite ſide, BC, of the triangle BAC remain conſtant, and the proportion between the *ſines* of angles and thoſe of their oppoſite ſides gives, $sin.\,C = \frac{sin.\,A}{sin.\,BC} \times sin.\,AB$, and, $sin.\,B = \frac{sin.\,A}{sin.\,BC} \times sin.\,AC$; it is evident that, the fluxions of *ſin.* C and *ſin.* B will be as thoſe of the *ſines* of the ſides, AB and AC, which are oppoſite to them. But, by *art.* 250, the fluxion of the arc which meaſures the angle C is to that of the arc AB as $\frac{\dot{\overline{sin.\,C}}}{cos.\,C}$ is to $\frac{\dot{\overline{sin.\,AB}}}{cos.\,AB}$, or, which is the ſame, $\dot{C} : \dot{\overline{AB}} :: \frac{\dot{\overline{sin.\,C}}}{cos.\,C} : \frac{\dot{\overline{sin.\,AB}}}{cos.\,AB}$: alſo, $\dot{\overline{sin.\,C}} : \dot{\overline{sin.\,AB}} :: sin.\,C : sin.\,AB$ (becauſe the relation of *ſin.* C : *ſin.* AB muſt be conſtant); wherefore, if *ſin.* C :

ſin. AB be ſubſtituted for $\dot{\overline{\textit{ſin}}}$. C : $\dot{\overline{\textit{ſin}}}$. AB in the two laſt terms of the preceding proportion, and afterwards the *tangents* into which they may be changed, we ſhall get, $\dot{C} : \dot{\overline{AB}}$: : *tang*. C : *tang*. AB. —— We might prove in the ſame manner that, $\dot{B} : \dot{\overline{AC}}$: : *tang*. B : *tang*. AC. Q. E. D.

COROLLARY I.

266. If A = 90°, we ſhall have; $\dot{C} : \dot{\overline{AB}}$: : R : *ſin*. AC, and alſo, $\dot{B} : \dot{\overline{AC}}$: : R : *ſin*. AB, *art*. 228.

COROLLARY II.

267. If the triangle be right-lined, the fluxions of the angles B and C will be equal; ſince the ſum of the three angles, as well as one of theſe angles, are conſtant quantities. Moreover, the *tangents* of the ſides, AB, AC, become then the ſides themſelves, and therefore we ſhall have; *As the fluxion of one of the angles is to that of its oppoſite ſide, ſo is the tangent of this angle to the ſame ſide*; that is, $\dot{C} : \dot{\overline{AB}}$: *tang*. C : AB, or, ſince $\dot{C} = \dot{B}$, $\dot{B} : \dot{\overline{AB}}$: : *tang*. C : AB; and, $\dot{C} : \dot{\overline{AC}}$: : *tang*. B : AC.

THEOREM VI.

268. *Suppoſing ſtill one angle with its oppoſite ſide conſtant; the fluxions of the other two ſides will be as the coſines of their oppoſite angles, and thoſe of the other two angles as the coſines of their oppoſite ſides*; that is, $\dot{\overline{AB}} : \dot{\overline{AC}}$: : *coſ*. C : *coſ*. B, and, $\dot{B} : \dot{C}$: : *coſ*. AC : *coſ*. AB.

DEMONSTRATION.

Fig. 28. Having taken D*b* = DB, and D*c* = DC, it is manifeſt that, the angles at *b* and *c* will be right, as alſo *bc* = $\beta\gamma$; and conſequently (by taking from theſe equal quantities the part βc, which is common),

mon), $\beta b = c\gamma$. This being premised, from the right-angled triangle βbB we shall get, Bβ : $b\beta$: : R : *cos*. B; and also from the right-angled triangle C$c\gamma$, $c\gamma$: Cγ : : *cos*. C : R; and therefore, by the multiplication of corresponding terms, Bβ : Cγ : : *cos*. C : *cos*. B; or, $\overline{\dot{AB}}$: $\overline{\dot{AC}}$: : *cos*. C : *cos*. B. Q. E. 1°. D.

2°. In the triangle DEF, whose parts are severally the supplements to those of the triangle BAC, we might find, $\overline{\dot{DF}}$: $\dot{EF}$: : *cos*. E : *cos*. D : wherefore, by substituting for each term its corresponding value in the triangle BAC, we shall get, $\dot{B}$: $\dot{C}$: : *cos*. AC : *cos*. AB. Q. E. 2°. D. *Fig.* 11.

COROLLARY I.

269. Since we have, $\overline{\dot{AB}}$: $\overline{\dot{AC}}$: : *cos*. C : *cos*. B, we shall also have (by *art*. 225) : : *cos*. AB × *sin*. AB × R — *cos*. AC × *cos*. BC × *sin*. AB : *cos*. AC × *sin*. AC × R — *cos*. AB × *cos*. BC × *sin*. AC : and, since we have, $\dot{C}$: $\dot{B}$: : *cos*. AB : *cos*. AC, we shall likewise have, by putting for *cos*. AC its value in *art*. 216, and dividing the two last terms by *cos*. AB; $\dot{C}$: $\dot{B}$: : R : $\frac{cos.\,B \times sin.\,BC \times tang.\,AB}{RR}$ + *cos*. BC.

COROLLARY II.

270. Since we have, by *Theo*. V. $\overline{\dot{AB}}$: $\dot{C}$: : *tang*. AB : *tang*. C, and, by the present Theorem, $\dot{C}$: $\dot{B}$: : *cos*. AB : *cos*. AC; if we multiply the corresponding terms of these two proportions together, we shall get, $\overline{\dot{AB}}$: $\dot{B}$: : R × *sin*. AB : *tang*. C × *cos*. AC. By a similar process we might find, $\overline{\dot{AC}}$: $\dot{C}$: : R × *sin*. AC : *tang*. B × *cos*. AB. Then, if we substitute in these two proportions for *sin*. AB and *sin*. AC, their respective values, $\frac{sin.\,AC \times sin.\,C}{sin.\,B}$

and

and $\frac{\textit{sin.}\ AB \times \textit{sin.}\ B}{\textit{sin.}\ C}$, the following analogies will arise; $\dot{\overline{AB}} : \dot{B} :: \textit{tang.}\ AC \times \textit{cos.}\ C : R \times \textit{sin.}\ B$, and, $\dot{\overline{AC}} : \dot{C} :: \textit{tang.}\ AB \times \textit{cos.}\ B : R \times \textit{sin.}\ C$. Likewise, if we again substitute, in the first proportion, $\frac{\textit{sin.}\ AC \times R}{\textit{tang.}\ AC}$ for *cos.* AC, we shall find, $\dot{\overline{AB}} : \dot{B} :: \textit{tang.}\ AC \times \textit{sin.}\ AB : \textit{tang.}\ C \times \textit{sin.}\ AC :: \textit{tang.}\ AC \times \textit{cos.}\ AB - \textit{sin.}\ AC \times \textit{cos.}\ BC : \frac{\textit{sin.}\ B \times \textit{sin.}\ AC \times \textit{sin.}\ BC}{R}$, by putting for *cos.* C its value, given in *art.* 225.—— Moreover, if we suppose the angle at A right, we shall get, from the latter proportion, $\dot{\overline{AC}} : \dot{C} :: \frac{1}{2}\textit{sin.}\ 2AC : R \times \textit{cot.}\ C$; by putting for *tang.* B and *cos.* AB, their values in *art.* 228, $\frac{\textit{cot.}\ C \times R}{\textit{cos.}\ BC}$ and $\frac{\textit{cos.}\ BC \times R}{\textit{cos.}\ AC}$, and for *sin.* AC × *cos.* AC the expression which is equal to it, by *art.* 14.

COROLLARY III.

271. If the triangle be right-lined, the fluxions of the sides will still be as the *cosines* of their opposite angles; and the fluxions of the angles as the *cosines* of their opposite sides. But, since in this case the sides are, or may be, considered as arcs indefinitely small, their *cosines* will be all equal, and therefore the fluxions of the angles also equal; a thing that we may be easily convinced of by other different considerations.

SECT. IV.

Concerning the fluxions of a spherical or right-lined triangle, when two of its sides remain constant.

THEOREM VII.

272. *The fluxion of the angle contained between the two constant sides is to the fluxion of either of the other*

two

two angles, as the product of the sine total and sine of the variable side is to the product of the sine of the side opposite to the latter angle and the cosine of the third angle adjacent to this side; that is, by supposing the sides AB and AC constant, $\dot{A} : \dot{B} :: R \times \textit{sin.}\,BC : \textit{sin.}\,AC \times \textit{cos.}\,C$, and, $\dot{A} : \dot{C} :: R \times \textit{sin.}\,BC : \textit{sin.}\,AB \times \textit{cos.}\,B$. *Fig.* 29.

DEMONSTRATION.

Supposing the angle BAC changed to BAc, let the sides, AC, Ac, be produced till they become quadrants, ACF, Acf; and, it is manifest that, Ff will be the measure of the variation of the angle at A: in like manner, let the sides, BC, Bc, be produced to, G, g, so that, BCG, Bcg, may be each 90°; and, it is again manifest that, Gg will be the measure of the variation of the angle at B. Moreover, let there be taken upon the side Bc the arc Bγ = BC, and there will by this means be formed the triangle Cγc, right angled at γ, wherein the angle γcC is perceptibly equal to the complement of the angle C. This being premised, on account of the similar sectors, Ff, Cc, we shall have, $Ff : Cc :: R : \textit{sin.}\,AC$; also, on account of the right-angled triangle Cγc, $Cc : C\gamma :: R : \textit{cos.}\,C$; and, on account of the similar sectors, Cg, Cγ, $C\gamma : Cg :: \textit{sin.}\,BC : R$; then, if we multiply the corresponding terms of these three proportions together, and substitute in the resulting product the values of Ff and Gg, we shall get, $\dot{A} : \dot{B} :: R \times \textit{sin.}\,BC : \textit{sin.}\,AC \times \textit{cos.}\,C$. —— We might prove by a similar process that, $\dot{A} : \dot{C} :: R \times \textit{sin.}\,BC : \textit{sin.}\,AB \times \textit{cos.}\,B$. Q.E.D.

COROLLARY I.

273. If in the first proportion we put $\frac{\textit{sin.}\,BC \times \textit{sin.}\,B}{\textit{sin.}\,A}$ for *sin.* AC, we shall find; $\dot{A} : \dot{B} :: R \times \textit{sin.}\,A :$ *sin.*

$\text{fin.}B \times \text{cof.}C :: \text{fin.}BC \times \text{tang.}C : \text{fin.}AC \times \text{fin.}C$ (by multiplying the two last terms by $\frac{\text{fin.}C}{\text{cof.}C}$) :: $\text{fin.}A \times \text{tang.}C : \text{fin.}B \times \text{fin.}C$ (by substituting for $\text{fin.}BC$ its value, $\frac{\text{fin.}AC \times \text{fin.}A}{\text{fin.}B}$) :: $\text{fin.}BC \times \text{tang.}C$: $\text{fin.}B \times \text{fin.}AB$ (by taking away $\text{fin.}C$ in one of the preceding comparisons) :: $\text{fin.}^2 BC : \text{cof.}AB \times R - \text{cof.}AC \times \text{cof.}BC$ (by putting in the first proportion the value of $\text{cof.}C$, given in *art.* 225) :: $\text{fin.}AB \times \text{fin.}A : \text{fin.}AC \times \frac{1}{2}\text{fin.}2C$ (by substituting in the same proportion $\frac{\text{fin.}AB \times \text{fin.}A}{\text{fin.}C}$ for $\text{fin.}BC$, and $\frac{1}{2}\text{fin.}2C$ for $\frac{\text{cof.}C \times \text{fin.}C}{R}$) :: $R^3 : \text{fin.}^2 B \times \text{cof.}AB - \text{fin.}B \times \text{cof.}B \times \text{cot.}A$ (by substituting $\frac{\text{fin.}BC \times \text{fin.}B}{\text{fin.}A}$ for $\text{fin.}AC$; $\text{cof.}AB \times \text{fin.}A \times \text{fin.}B - \text{cof.}A \times \text{cof.}B \times R$ (*art.* 221) for $\text{cof.}C$, and $\text{cot.}A$ for $\frac{R \times \text{cof.}A}{\text{fin.}A}$) :: $R^3 : \text{cof.}AB \times RR - \text{fin.}AB \times \text{cof.}B \times \text{cot.}BC$; by taking away $\text{cot.}A$ (*art.* 214), and putting R^2 for $\text{fin.}^2 B + \text{cof.}^2 B$.

COROLLARY II.

274. Similar substitutions in the second proportion will give the following comparisons; $\dot{A} : \dot{C} :: R \times \text{fin.}A : \text{fin.}C \times \text{cof.}B :: \text{fin.}BC \times \text{tang.}B : \text{fin.}AB \times \text{fin.}B :: \text{tang.}B \times \text{fin.}BC : \text{fin.}AC \times \text{fin.}C :: \text{tang.}B \times \text{fin.}A : \text{fin.}B \times \text{fin.}C :: \text{fin.}^2 BC :: \text{cof.}AC \times R - \text{cof.}AB \times \text{cof.}BC$.

COROLLARY III.

275. If the triangle be right-lined, the *sines* of the sides will then become the sides themselves, and therefore we shall have; $\dot{A} : \dot{B} :: R \times BC : AC \times \text{cof.}C$, and, $\dot{A} : \dot{C} :: R \times BC : AB \times \text{cof.}B$; that is to say, *The fluxions of the angle included between the constant sides and either of the other two angles, will*

will be in the direct compound ratio of the sides opposite to these angles and of the radius to the cosine of the third angle.

THEOREM VIII.

276. *The fluxion of the angle contained between the constant sides is to that of its opposite side, as radius squared is to the rectangle under the sine of either of the other angles and that of its adjacent side*; that is, $\dot{A} : \dot{\overline{BC}}$ :: R^2 : *sin.* C × *sin.* AC :: R^2 : *sin.* B × *sin.* AB. *Fig.* 25.

DEMONSTRATION.

The similar sectors, F*f*, C*c*, give (as we have already observed), F*f* : C*c* :: R : *sin.* AC; and the little right-angled triangle Cγc, C*c* : γc :: R : *sin.* C; wherefore, by multiplying corresponding terms together, we shall get, F*f* : γc :: R^2 : *sin.* AC × *sin.* C; or, $\dot{A} : \dot{\overline{BC}}$ :: R^2 : *sin.* AC × *sin.* C.—— By supposing the angle A to flow towards B, we should find in like manner, $\dot{A} : \dot{\overline{BC}}$ :: R^2 : *sin.* AB × *sin.* B. Q. E. D.

COROLLARY I.

277. If we substitute $\frac{sin.BC \times sin.B}{sin.A}$ and $\frac{sin.AB \times sin.B}{sin.C}$ for *sin.* AC, we shall get; $\dot{A} : \dot{\overline{BC}}$ :: R^2 × *sin.* A : *sin.* BC × *sin.* B × *sin.* C :: R^2 × *sin.* C : *sin.* AB × *sin.* B × *sin.* C :: R^2 × *sin.* BC × *sin.* C : *sin.*^{2} AB × *sin.* A × *sin.* B, by putting $\frac{sin.\ AB \times sin.\ A}{sin.\ BC}$ for *sin.* C.

COROLLARY II.

278. If the triangle be right-lined, we shall have; $\dot{A} : \dot{\overline{BC}}$ :: R^2 : AC × *sin.* C :: R^2 : AB × *sin.* B :: R : AD (by supposing AD to be a perpendicular let fall from the angle A upon the opposite side, BC): whence we infer that, if two sides of a right-lined triangle be constant, *The fluxion of the*

the angle included between these sides will be to that of its opposite side, as radius is to the perpendicular let fall from this angle upon the said side.

THEOREM IX.

279. *The fluxion of either of the angles adjacent to the variable side is to the fluxion of this side, as the cotangent of the other adjacent angle is to the sine of the*

Fig. 29. *said side*; that is, $\dot{C} : \dot{\overline{BC}} :: cot.\ B : sin.\ BC$, and, $\dot{B} : \dot{\overline{BC}} :: cot.\ C : sin.\ BC$.

DEMONSTRATION.

Since we have, by *art.* 272, $\dot{A} : \dot{C} :: R \times sin.BC : sin.\ AB \times cos.\ B$, and, by *art.*276, $\dot{\overline{BC}} : \dot{A} :: sin.\ C \times sin.\ AC : R^2$; if we multiply these two proportions together, we shall get, $\dot{\overline{BC}} : \dot{C} :: sin.\ BC \times sin.\ C \times sin.\ AC : R \times sin.\ AB \times cos.\ B :: sin.\ BC : \frac{sin.\ AB \times cos.\ B \times R}{sin.\ C \times sin.\ AC} :: sin.\ BC : cot.\ B$; by substituting $\frac{sin.\ AC \times sin.\ C}{sin.\ B}$ for $sin.\ AB$, and $cot.\ B$ for $\frac{cos.\ B \times R}{sin.\ B}$; and consequently, $\dot{\overline{BC}} : \dot{C} :: sin.\ BC : cot.\ B$, or, *invertendo*, $\dot{C} : \dot{\overline{BC}} :: cot.\ B : sin.\ BC$. Q. E. D.

COROLLARY I.

280. Since, $\dot{B} : \dot{\overline{BC}} :: cot.\ C : sin.\ BC$, it will be easy to obtain the following comparisons; $:: R \times cos.\ C : sin.\ AB \times sin.\ A$ (by putting $\frac{sin.\ A \times sin.\ AB}{sin.\ C}$ for $sin.\ BC$, &c.) $:: RR : tang.\ C \times sin.\ BC$ (by putting $\frac{R^2}{tang.\ C}$ for $cot.\ C$) $:: cot.\ C \times sin.\ B : sin.AC \times sin.\ A$ (by substituting $\frac{sin.\ AC \times sin.\ A}{sin.B}$ for $sin.\ BC$) $:: R : \frac{sin.BC \times sin.\ B \times R}{cot.\ AB \times sin.\ BC - cos.\ B \times cos.\ BC}$ (by *art.* 214) $:: cot.\ AB \times sin.\ BC - cos.\ B \times cos.\ BC : sin.\ BC \times$

sin.

sin. B : : $\frac{cot.\ AB \times R}{sin.\ B} - \frac{cot.\ BC \times R}{tang.\ B}$: R (by dividing by *sin.* BC × *sin.* B, *&c.*) : : *cot.* AB — $\frac{cos.\ B \times cot.\ BC}{R}$: *sin.* B, by multiplying by *sin.* B, *&c.* —— We might find comparisons exactly similar for the proportion, $\dot{C} : \overline{\dot{BC}}$: : *cot.* B : *sin.* BC; but think it unnecessary to particularize them.

Corollary II.

281. Since, $\dot{B} : \overline{\dot{BC}}$: : *cot.* C : *sin.* BC, and, $\overline{\dot{BC}}$: $\dot{C}$: : *sin.* BC : *cot.* B; if we multiply the corresponding terms of these two proportions together, we shall find, $\dot{B} : \dot{C}$: : *tang.* B : *tang.* C : : $\frac{sin.\ B}{cos.\ B}$: $\frac{sin.\ C}{cos.\ C}$: : *sin.* B × *cos.* C : *sin.* C × *cos.* B : : *sin.* AC × *cos.* C : *sin.* AB × *cos.* B, by substituting $\frac{sin.\ AB \times sin.\ B}{sin.\ AC}$ for *sin.* C.

Corollary III.

282. If the triangle be right-lined, the Theorem will give, $\overline{\dot{BC}} : \dot{C}$: : BC : *cot.* B : : BC × *sin.* B : *cos.* B × R; or, as the perpendicular let fall from the angle C upon the side AB is to the *cosine* of the angle B. In like manner, $\overline{\dot{BC}}$ *will be* : $\dot{B}$: : BC : *cot.* C; and lastly, $\dot{B} : \dot{C}$: : *tang.* B : *tang.* C.

SECT. V.

Concerning the fluxions of any spherical Triangle, whereof two of the angles are supposed constant.

Theorem X.

283. *Supposing the two angles*, B, C, *in the spherical triangle* BAC *constant*; *the fluxion of the side to which these angles are adjacent will be to that of either* *Fig.* 28.

 of

of the other sides, in the direct compound ratio of the sines of the angles opposite to these sides and of the radius to the cosine of the third side; that is, $\dot{\overline{BC}} : \dot{\overline{AB}}$: : *sin.* A × R : *sin.* C × *cos.* AC.

DEMONSTRATION.

Fig. 11. In the triangle DEF, which has all its parts supplemental to those of the triangle BAC, the two sides, DF, FE, will be variable: whence this case appears to be reduced to that of the first Theorem of the last Section; therefore, we shall have, $\dot{F} : \dot{D}$: : R × *sin.* DE : *sin.* FE × *cos.* E; and, by taking the corresponding values in the triangle BAC, $\dot{\overline{BC}} : \dot{\overline{AB}}$: : R × *sin.* A : *sin.* C × *cos.* AC. —— We might prove in like manner that, $\dot{\overline{BC}} : \dot{\overline{AC}}$: : R × *sin.* A : *sin.* B × *cos.* AB. Q. E. D.

THEOREM XI.

284. *The fluxion of the side contained between the constant angles is to that of the opposite angle, as the cosecant of either of the other sides is to the sine of the constant angle above this side*; that is, $\dot{\overline{BC}} : \dot{A}$: : *cosec.* AB : *sin.* B : : *cosec.* AC : *sin.* C.

DEMONSTRATION.

It is well known, that the *cosecant* of an arc is equal to the square of the radius divided by the *sine* of this arc; therefore, the whole is reduced to prove that, $\dot{\overline{BC}} : \dot{A}$: : R^2 : *sin.* B × *sin.* AB; a thing easily done by means of the triangle DEF and *Theo.* VIII. by pursuing a method similar to that used in the demonstration of the last Theorem. Q. E. D.

THEOREM XII.

285. *The fluxion of either of the sides opposite to the constant angles is to the fluxion of the third angle, as the cotangent of the other side is to the sine of the said angle*;

angle; that is, $\dot{\overline{AB}} : \dot{A} :: cot.\,AC : ſin.\,A$, and, $\dot{\overline{AC}} : \dot{A} :: cot.\,AB : ſin.\,A$.

DEMONSTRATION.

This propoſition may be immediately deduced from *Theo.* IX. by applying it to the triangle DEF, and making the neceſſary changes. Q. E. D.

COROLLARY I.

286. It follows from the laſt Theorem, that the fluxions of the ſides, AB, AC, oppoſite to the conſtant angles, are as the *tangents* of theſe ſides.

COROLLARY II.

287. By ſuppoſing the triangle right-lined, the firſt Theorem ſhews us, that the fluxions of the ſides are to each other as the *ſines* of the angles oppoſite to theſe ſides; which thing is alſo well known by, and eaſily proved from, the elements of Geometry. From the ſecond Theorem nothing can be deduced, ſince in a plain triangle, which is ſuppoſed to have two of its angles conſtant, $\dot{A}$ cannot obtain. The ſame aſſertion may be made with reſpect to the third Theorem. Laſtly, from the preceding Corollary it may be inferred, that the fluxions of the ſides are as the ſides themſelves; a thing likewiſe demonſtrable by the elements of plain Geometry.

OBSERVATION.

288. The foregoing Theorems might have been eaſily demonſtrated from the figures uſed in the projections, by finding the evaneſcent relations of the different lines therein deſcribed. They might likewiſe have been deduced from the formulæ which have been obtained by the application of Algebra, by ſuppoſing ſome of the letters, *a*, *b*, *c*, *d*, *f*, *g*, &c. variable, according to the conditions

of

of the Theorem to be demonſtrated, and then paſſing from the fluxions of the *ſines*, *tangents*, &c. to thoſe of the arcs of a circle: but the method of Mr. *Cotes* appeared to me the cleareſt, and at the ſame time the eaſieſt; for which reaſon I have not given much more than a tranſlation of his excellent little Treatiſe, " *De æſtimatione errorum in mixtâ* " *Matheſi:*" the theory indeed I have attempted to render ſomewhat more ſimple, by applying it to plain triangles in bare Corollaries; but ſuch a liberty I hope my readers will candidly excuſe.—— What now remains is to make ſome applications of this theory to different examples.

Example I.

Fig. 30. 289. *Let there be an object whoſe height*, AB, *we would meaſure by taking with an inſtrument the angle* ACB; *it is required to determine what error may be committed with reſpect to the ſaid height, by ſuppoſing the error in the meaſure of the obſerved angle to be known.*

SOLUTION.

It is manifeſt that, in the triangle BAC, the ſide AC and right angle, A, remain conſtant; therefore, by *art.* 258, we ſhall have, $\dot{\overline{AB}} : \dot{C} :: BC : \textit{ſin.}\ ABC :: 2AB : \textit{ſin.}\ 2ACB$ (for by ſubſtituting for *ſin.* 2ACB its value $\frac{2\,\textit{ſin.}\ ACB \times \textit{ſin.}\ ABC}{R}$, by *art.* 13, this proportion becomes, $BC : R :: AB : \textit{ſin.}\ ACB$); and conſequently, *The error of the height* AB *is to the meaſure of the error of the angle* C, *as the double of this height is to the ſine of the double of the obſerved angle.* The error therefore, which can prevail in the determination of the ſaid height, will be the leaſt poſſible, when the *ſine* of double the obſerved angle is the greateſt poſſible; that is, when this angle is 45°: for which reaſon, when we would find the height, AB, of any object, by the obſervation

of

of any angle, as C, we muſt ſo contrive it, that the angle obſerved may be the neareſt 45° poſſible. Q. E. I.

290. Let us now proceed to diſcover what the error of the height AB may be, upon a ſuppoſition that a miſtake of one minute is committed in the determination of the angle C. If we make the radius = 10000000, the arc of a minute, which meaſures the ſuppoſed error, will be 2909; the double whereof is 5818: but this quantity is to the *ſine* of double the angle of 45° or to radius, as 1 to 1719; and therefore, according to this ſuppoſition, the miſtake concerning the ſaid height AB will be $\frac{1}{1719}$th part of the height itſelf. —— If the error of the angle, ACB, be increaſed or diminiſhed, the error of the height will be increaſed or diminiſhed in the ſame ratio: alſo, if the angle obſerved be greater or leſs than 45°, the error of the height will be increaſed in the ratio of radius to the *ſine* of double the ſaid angle.

EXAMPLE II.

291. *It is required to find the hour of the day or night, by obſerving the altitude of a ſtar; and alſo to aſſign the error of time, by ſuppoſing the error in the obſerved altitude to be known.*

SOLUTION.

Let there be a ſpherical triangle PZS, wherein P is the pole, Z the zenith of the place of obſervation, and S the place of the ſtar obſerved; then will PS be the complement of the ſtar's declination; ZS the complement of its altitude; PZ the complement of the latitude; and the angle ZPS a variable hour-angle contained betwixt the conſtant ſides, PZ, PS. This being premiſed, the fluxion of the hour-angle SPZ is (by *art.* 276) to the fluxion of the oppoſite ſide SZ, as the *coſecant* of the *Fig.* 31.

the angle PZS is to the *sine* of the side PZ; or, $\dot{P} : \overline{\dot{ZS}} :: R^2 : \textit{sin.}\ Z \times \textit{sin.}\ PZ$: but the variation of the angle at P is the measure of the error in time; therefore, by taking $\dot{P}$ for the said error, we shall get, $\dot{P}$ or $\dot{t} = \frac{R^2 \times \overline{\dot{ZS}}}{\textit{sin.}Z \times \textit{sin.}PZ}$. Q. E. I.

COROLLARY.

292. It follows from hence that, if we suppose the same error to prevail in different observations, and the latitude of the place to continue also the same, the error of time will not be altered, whatever the altitude of the star in the given vertical be; since $\frac{R^2}{\textit{sin.}\ Z}$ will be always a constant quantity. Moreover, we may perceive that, by still supposing the latitude constant, the error of time will be the least, when the star is observed on the prime vertical: and, if we assume two different latitudes, the error will be the least possible, when the observation is made at the equator, and the star situate on the prime vertical. For instance (granting the truth of these two assertions); if we suppose a mistake of one minute to be committed in the observation of the altitude, we shall find, by making the calculation, that the error in time is $4''$: but, if we again suppose the observer to be removed to a certain distance from the equator, the error then (provided the star be still observed on the prime vertical) will be to that aforegoing, in the ratio of radius to the *cosine* of the latitude; so that for a latitude of $45°$, the error will be $5''\frac{2}{3}$; for a latitude of $50°$, $6''\frac{2}{9}$, and for a latitude of $55°$, $6''\frac{37}{38}$. Lastly, if a star be observed on any vertical inclined to the meridian, the error will be yet increased with regard to those above given, in the ratio of radius to the *sine* of the angle contained between them; and

and its variations at the ſame time depend on thoſe ſuppoſed in the firſt obſervation *.

EXAMPLE III.

293. *Let*, LSK, L′S′K′, *be two almacanters or circles parallel to the horizon, and*, S, S′, *the points where a ſtar (whoſe declination is ſuppoſed the ſame during the whole day) paſſes over theſe circles; it is required to find what the relation of the angles*, PSZ, PS′Z, *made by the hour-circles*, PS, PS′, *with their correſponding verticals, ought to be, ſo that the time taken up by the ſtar in paſſing from one almacanter to the other may be the leaſt poſſible.* Fig. 32.

SOLUTION.

From the *data* of the Problem it is manifeſt that, in the triangles, PZS, PZS′, the ſides, PZ, ZS and ZS′, remain conſtant, whilſt the ſides, PS, PS′, vary. Let then h and h' be put for the hour-angles at P; S and S′ for the angles at the ſtar at S and S′, made by the hour-circles PS and PS′ with the correſponding verticals ZS and ZS′, and let D repreſent the declination; the fluxion whereof, $\dot{D}$, will be negative, ſince, if the declination be northerly (as we have at preſent ſuppoſed it), the arcs, PS, PS′, will decreaſe as the declination increaſes. This being premiſed, we ſhall have, by *art.* 279, $\dot{h} = -\frac{cot.\ S \times \dot{D}}{cof.\ D}$, and, in like manner, $\dot{h}' = -\frac{cot.\ S' \times \dot{D}}{cof.\ D}$: then, ſince the time over the arc SS′ is to be a *minimum*, the difference of theſe fluxions of the hour-angles muſt be equal to nothing; or, $\frac{cot.\ S \times \dot{D}}{cof.\ D} - \frac{cot.\ S' \times \dot{D}}{cof.\ D} = 0$: whence we immediately deduce, $cot.\ S = cot.\ S$; and therefore,

* For theſe two Examples ſee *Cotes*'s aforementioned Treatiſe.

fore, $S = S'$. Now, if this ſtar be the ſun, and one of the almacanters coincide with the horizon, whilſt the other becomes the crepuſcular circle (which is uſually ſuppoſed to be 18° below the horizon), the conſequence juſt inferred intimates to us, that, on a day when the thing required happens, the angles at the ſun, both at the horizon and crepuſcular circle, are equal. From this conſequence therefore, the day of ſhorteſt twilight may be found, provided we firſt ſeek what the declination anſwering to this determinate relation of the angles at the ſun ought to be: which we proceed to do in the following

PROBLEM.

294. *Suppoſing the angles, made by two hour-circles with their correſponding verticals, equal; it is required to find the ſun's declination anſwering thereto.*

SOLUTION.

Fig. 32. Let S and S′ repreſent the angles contained between the hour-circles and their correſponding verticals; C and C′ the diſtances of two almacanters from the horizon; L the latitude of the place, and D the ſun's declination: then (by *art.* 225) we ſhall get from the triangle PZS, $\frac{cof.\ D \times cof.\ S}{R} = \frac{R \times fin.\ L - fin.\ C \times fin.\ D}{cof.\ C}$, and $\frac{cof.\ D \times cof.\ S'}{R} = \frac{R \times fin.\ L + fin.\ C' \times fin.\ D}{cof.\ C'}$ from the triangle PZS′; putting + before *fin.* C′ in the ſecond formula, becauſe ZS′, as well as its oppoſite ſide, is obtuſe. Now the firſt members of theſe equations being equal, the ſecond will be ſo likewiſe, and give this new equation, $R \times fin.\ L \times cof.\ C' - fin.\ C \times fin.\ D \times cof.\ C' = R \times fin.\ L \times cof.\ C + fin.\ C' \times fin.\ D \times cof.\ C$: whence we immediately deduce, $fin.\ C' \times cof.\ C + fin.\ C \times cof.\ C' : R \times \overline{cof.\ C' - cof.\ C} ::$ *fin.*

ſin. L : *ſin.* D; or, by *art.* 85, *ſin.* $\overline{C'+C}$: *coſ.* C′ — *coſ.* C : : *ſin.* L : *ſin.* D, or otherwiſe, by *art.* 87 and 94, *coſ.* $\frac{C+C'}{2}$: *ſin.* $\frac{C-C'}{2}$: : *ſin.* L : *ſin.* D. Then, if we ſuppoſe C = 0 (as it really muſt be in the caſe of the ſhorteſt twilight), we ſhall get this laſt proportion; R : — *tang.* $\frac{C'}{2}$: : *ſin.* L : *ſin.* D; that is to ſay, *As radius is to the tangent of half the depreſſion of the crepuſcular circle below the horizon, ſo is the ſine of the latitude to the ſine of the ſun's declination:* where obſerve, that, as the declination is negative (becauſe *tang.* $\frac{C'}{2}$ is preceded by —), it muſt be of a denomination contrary to the latitude of the place.

OLD METHOD *of* SOLUTION.

295. Let u be put for the *coſine* of the hour-angle at the inſtant when the twilight ends, and the angle itſelf will be repreſented by, $\int \times \frac{r\dot{u}}{\sqrt{r^2-u^2}}$; alſo, let z be put for the *coſine* of the ſemi-diurnal arc, and this arc will be, $\int \times \frac{r\dot{z}}{\sqrt{r^2-z^2}}$: then muſt $\frac{\dot{u}}{\sqrt{r^2-u^2}} - \frac{\dot{z}}{\sqrt{r^2-z^2}} = 0$. Moreover, let c be put for the *ſine* of the latitude, and g for its *coſine*; h for the *ſine* of the depreſſion of the crepuſcular circle below the horizon, and y for the *ſine* of the ſun's declination; and we ſhall have, $u = \frac{hr^2 - cry}{g\sqrt{r^2-y^2}}$, and, $z = \frac{cry}{g\sqrt{r^2-y^2}}$: then, if by means of theſe values we exterminate u and z in the firſt equation, and afterwards divide by $\dot{y}$, we ſhall get, $hy^4 + 2cry^3 - g^2hy^2 - 2cr^3y - c^2hr^2 = 0$; whence, y

$= r$;

$= r$; $- r$; $\frac{-cr - c\sqrt{r^2 - h^2}}{h}$, and, $\frac{-cr + c\sqrt{r^2 - h^2}}{h}$.
But of these (four) values of y, the two first, indicating the sun to be at the pole (where it can never physically arrive), are foreign to the question and unapplicable: of the other two, the greatest gives the *minimum* of the sum of the semi-diurnal arc, and the arc which the sun passes over from noon to the end of twilight; whilst the least gives the *minimum* of the difference of these arcs: consequently, the declination required hath for its *sine*, $y = \frac{-cr + c\sqrt{r^2 - h^2}}{h}$. Therefore, if we put t for the *tangent* of half the depression of the crepuscular circle below the horizon, we shall have, by *art.* 75, $y = \frac{-ct}{r}$, or, $r : t :: c : -y$; which shews, as before, that the sun's declination is of a denomination contrary to the latitude of the place*.

OBSERVATION.

296. From the first of the two preceding solutions it follows that, if any two spherical triangles have two sides equal each to each, and also an equal angle, opposite to one of the equal sides, we shall have this analogy; *As the cosine of the side adjacent to the equal angle is to the cosine of the side opposite to the same angle, so is the cosine of half the sum of the other two sides to the cosine of half their difference.* —— This proportion is immediately deduced from, *cos.* $\frac{C+C'}{2}$: *sin.* $\frac{C-C'}{2}$ :: *sin.* L : *sin.* D; by observing that the particular denominations in the Problem will be rendered general, if such things only be preserved as are essential to the triangles we have occasion to consider.

If,

* Should not this solution appear entirely perspicuous, consult *Maupertuis*'s Astronomie Nautique, p. 38, &c.

If, in like manner, it was required to find what the declination of a ſtar ought to be, ſo that, in paſſing over the interval contained between two given hour-circles, its change in altitude might be the *greateſt* poſſible; we ſhould get by a ſimilar proceſs, and with the preceding denominations, $Coſ.\ \frac{h'-h}{2} : coſ.\ \frac{h'+h}{2} :: tang.\ \mathrm{L} : tang.\ \mathrm{D}$ *.

EXAMPLE IV.

297. *To find at all times the correction neceſſary to be made for noon-day, as deduced from correſponding altitudes of the ſun; whoſe declination may be ſuppoſed to undergo a little change during the interval of two equal altitudes.*

SOLUTION.

Aſtronomers (in order to be certified of the inſtant of noon) take the altitude of the ſun's centre ſome time in the morning; then obſerve at what time in the afternoon it comes to the ſame altitude, and afterwards take half the interval of time elapſed between theſe two obſervations, as marked by a clock, for the inſtant of the ſun's paſſage over the meridian. Now this method would be quite exact and rigorous, provided the ſun did not change its declination during the interval of the obſervations; but this is a thing never ſtrictly true. At the time of the ſolſtices this error is the leaſt poſſible, becauſe then the ſun does not ſenſibly change its declination for two or three days; but at all other times of the year the operation abovementioned wants a correction, which it is required by this Problem to determine: for, it is evident

* To me it appears that, as ZS′ and the angle ZPS′ are obtuſe, the proportion ought to be, $Sin.\ \frac{h+h'}{2} : ſin.\ \frac{h-h'}{2} :: tang.\ \mathrm{L} : tang.\ \mathrm{D}$.

evident that, the ſun when in the aſcending ſigns arrives later at the ſame altitude, and ſooner when in the deſcending ſigns; and therefore, in the ſix firſt months ſomething muſt be ſubtracted, and in the ſix other months ſomething added, that the preciſe inſtant of noon-day may be obtained. —— Now to find this correction; let P*s* be an hour-circle very near to PS, and terminated by the ſame almacanter LSK: then, it is manifeſt that, the angle SP*s*, reduced into time, will expreſs how much later, or ſooner, the ſun arrives at the ſame altitude in the afternoon than in the morning; and conſequently, the half thereof will be the time which muſt be ſubtracted from, or added to, the inſtant of noon deduced from correſponding altitudes. This being premiſed, if the change in declination be ſuppoſed ſo ſmall, as to coincide with the naſcent variation or fluxion of the arc PS; it is evident that, the ſides, PZ, ZS, of the triangle PZS will remain conſtant, whilſt the ſide PS varies. But in *art*. 280 we inferred that, $\dot{B} : \dot{\overline{BC}} :: \frac{cot.\,AB}{ſin.\,B} \pm \frac{cot.\,BC}{tang.\,B} : R$; and therefore, by ſuppoſing the points, B, A, C, to repreſent, P, Z, S, we ſhall have, $\dot{P} = \frac{\dot{\overline{PS}}}{R} \times \overline{\frac{tang.\,lat.}{ſin.\,P} \pm \frac{tang.\,dec.}{tang.\,P}}$; a formula, exactly agreeing with that which Mr. *Maupertuis* hath given in p. 34 of his *Aſtronomie Nautique*: where obſerve, that the ſign + prevails, when the declination is ſoutherly, but the ſign —, when it is northerly; as we may be eaſily convinced from the ſolutions given in the preceding Chapter.—— The half of this expreſſion, reduced into time, will be the correction required *. Q. E. I.

They,

* See *Robertſon*'s Tranſlation of *De la Caille*'s Aſtronomy, p. 150.

They, who are desirous of seeing a variety of other applications of fluxional analogies, may have recourse to the *Memoirs of the Academy* for the year 1744, or to the *Astronomie* of Mr. *De la Lande*. It may be proper, however, to observe, that several of the solutions obtained by this means are not attended with a truly Geometrical strictness; but approach so much the nearer to precision, as the arcs under consideration approach the nearer to nascent arcs. —— We shall now conclude this Chapter with the solution of a Problem which is more curious than useful (but may nevertheless intimate to us some important truths, by the simplicity of the solution whereat we arrive), viz. with finding the area of any spherical triangle. Several Geometers, amongst whom may be reckoned Mr. *J. Bernoulli*, have investigated the quadrature of this portion of the surface of the sphere; taking for the element (indefinitely small part) thereof a differential (fluxion), whose integral (fluent) depends on the quadrature of the hyperbola. The subjoined solution, which we have in a great measure deduced from the Works of Dr. *Wallis*, is indisputably one of the most elegant that can be given, and may furnish us with methods of obtaining, by the quadrature of the circle, the fluents of several fluxions which appear very complicated.

PROBLEM.

298. *To find the area of any spherical triangle*, BAC. *Fig.* 33.

SOLUTION.

In the first place, let the sides, AB, BC, containing any angle, B, be produced, so that, BAb, BCb, ABa, CBc, may be arcs of 180°; and, it is manifest that, the triangles, AbC, aBc, will be equal in all respects. In the next place, let a plane be conceived to pass through the points, A, C; a, c, and

and of conſequence through the centre of the ſphere; and the ſurface $ACBac$ will be equal to that of an hemiſphere; which we ſhall denote by s, putting the radius of its great circle $= r$, and circumference $= c$. This done, if the triangles, ABC, CBa, aBc, cBA and AbC, be repreſented by, M, O, N, P and n, reſpectively, the portion of the ſpherical ſurface $bABCb$ will be determined by this analogy, $c : B :: 2s : \frac{2sB}{c} = M + n$ or $M + N$; the portion $ACaBA$ by, $c : A :: 2s : \frac{2sA}{c} = M + O$, and that of $CBcAC$ by, $c : C :: 2s : \frac{2sC}{c} = M + P$: then, if we collect theſe three equalities, we ſhall have, $\overline{B + A + C} \times \frac{2s}{c} = 3M + N + O + P$; whence, as $M + N + O + P = s$, M, or the area of the triangle BAC, will be found $= \overline{B + A + C} \times \frac{s}{c} - \frac{1}{2}s$; or, as the ſurface of an hemiſphere $= cr$, it will be otherwiſe found $= \overline{A + B + C} \times r - \frac{1}{2}cr = \overline{A + B + C - \frac{1}{2}c} \times r$; and therefore, if from the ſum of the three angles of any ſpherical triangle 180° be ſubtracted, and the remainder multiplied by the radius ($= 57^\circ.2957795$), the product will give the ſurface of ſuch triangle in ſquare degrees *.

* See *Emerſon*'s Trigonometry, p. 187, 2d edition.

CHAP.

CHAP. VI.

Application of the formulæ demonſtrated in the ſecond Chapter to ſeveral Problems of Aſtronomy.

THough, by the Tables annexed to the ſecond Chapter, our readers might be enabled to ſolve all the Problems, which can be reduced to any of the caſes of right or oblique-angled ſpherical triangles; yet, as they, who are not well verſed in the performance of trigonometrical calculations, might be afraid of committing miſtakes upon applying the rules therein given to the Logarithms, we thought ourſelves under an indiſpenſible obligation of annexing a proper number of examples; in order to render this little Treatiſe as complete as poſſible. We might, like all other Writers upon Trigonometry, have made the applications to triangles, whoſe parts ſhould have had no particular denomination; but we judged it better to conſider the ſeveral triangles as relating to the circles of the ſphere, that the ſolutions thence derived might become more intereſting to Learners.——We will ſuppoſe then throughout this Chapter the principal circles of the ſphere to be denoted, as they are exhibited in *fig.* 34, viz. HOR to be the horizon; AOQ the equator; IEL the ecliptic, making with the equator an angle that diminiſhes very ſlowly, and which, with moſt other Aſtronomers, we will here imagine to be $23^{\circ}\ 28'\ 30''$; and PZH the meridian, wherein, P, *p*, repreſent the poles, viz. P the pole which is elevated above the horizon, and *p* that depreſſed below. The altitude of the pole we will ſuppoſe to be $40^{\circ}\ 51'$, as it is for the City of *Paris*; and, in the laſt place, denote any ſtar or its place by S.——We take it for

granted that the reader is perfectly acquainted with the meaning of a star's declination, right ascension, longitude, latitude, &c. as well as with all the principal circles of the sphere; abundant explanations whereof may be met with in a variety of Treatises on that subject.

PROBLEM I.

Fig. 34. 299. *Given the sun's place in the ecliptic; to find his declination, or distance,* SD, *from the equator.*

SOLUTION.

Let, S, the sun's place, be 18° 24′ of Taurus: then the arc ES will be 48° 24′; and therefore in the triangle SDE, right angled at D, we know the hypothenuse, ES, and the angle DES = 23° 28′ 30″ = the obliquity of the ecliptic; whence we shall get by the Table for right-angled triangles,

$$\textit{sin.}\,DS = \frac{\textit{sin.}\,ES \times \textit{sin.}\,DES}{R}.$$

LOGARITHMICALLY.

9.873784 = *log. sin.* 48° 24′.
9.600263 = *log. sin.* 23° 28′ 30″.
9.474047 = *log. sin.* 17° 19′ 49″ = DS.

PROBLEM II.

300. *Given the sun's declination, as also the season of the year; to determine his place in the ecliptic.*

SOLUTION.

Let us suppose the sun's declination to be 17° 19′ 49″ northerly, and the season to be the spring: then we must find the arc, ES, of the ecliptic contained between the sun and first degree of Aries. Now to do this, it is manifest that, in the right-angled triangle DES, we shall know the angle E with its opposite side, DS; and therefore the hypothenuse, ES, will be obtained by the common analogy, which gives, $\textit{sin.}\,ES = \frac{\textit{sin.}\,DS \times R}{\textit{sin.}\,DES}$.

LOGA-

LOGARITHMICALLY.

9.474047 = *log. sin.* DS = 17° 19′ 49″.
0.399737 = *arith. comp. sin.* 23° 28′ 30″.
9.873784 = *log. sin.* ES = 48° 24′ 0″.

SCHOLIUM.

301. From the preceding Problem the method of computing Tables for the sun's motion in the ecliptic and his variations (by means of observations) will readily appear. For, the latitude of the place of the observer being accurately determined, and a quadrant fixed in the plane of the meridian, the observation of the sun's meridian altitude gives his declination, or distance from the equator; from whence his longitude for any day is easily obtained, by the method above specified.

PROBLEM III.

302. *Given the sun's place in the ecliptic; to find his right ascension, or* (which is the same) *the point,* D, *of the equator which passes over the meridian at the same time with the sun.*

SOLUTION.

Supposing still the arc ES to be 48° 24′, it is required to find the arc ED. Now the Table of right-angled triangles gives for this case, *tang.* ED $= \frac{tang.\ ES \times cos.\ DES}{R}$.

LOGARITHMICALLY.

10.051664 = *log. tang.* 48° 24′ 0″.
9.962480 = *log. cos.* 23° 28′ 30″.
10.014144 = *log. tang.* 45° 55′ 58″.

PROBLEM IV.

303. *Given the latitude of a place, and the sun's declination; to find his ortive or occasive amplitude, or at what distance from the true east or west he rises or sets.*

 SOLU-

SOLUTION.

Supposing always the latitude to be 48° 51′, and that the sun's declination is 18° 30′ northerly; it is manifest that, all we have to do is to find the hypothenuse of a right-angled triangle, DOS, wherein the angle DOS = the complement of the latitude, and the side DS = the sun's declination, are given. We shall therefore have by the common analogy, $\textit{sin. } OS = \frac{\textit{sin. } DS \times R}{\textit{sin. } DOS}$.

LOGARITHMICALLY.

9.501476 = *log. sin.* DS = 18° 30′ 0″.
0.181753 = *arith. com. log. sin.* 41° 9′ 0″.
9,683229 = *log. sin.* OS = 28° 49′ 45″.

COROLLARY I.

304. From this Problem it follows that, if the sun's declination and amplitude be given, the latitude of the place will be easily obtained: for, according to this supposition, in the right-angled triangle DOS, we shall know (beside the right angle) the hypothenuse and leg DS; wherefore we shall have, $\textit{sin. } DOS = \frac{\textit{sin. } DS \times R}{\textit{sin. } OS}$.

LOGARITHMICALLY.

9.501476 = *log. sin.* DS = 18° 30′ 0″.
0.316771 = *arith. comp. log. sin.* 28° 49′ 45″.
9.818247 = *log. sin.* 41° 9′; *whence the lat.* = 48° 51′.

COROLLARY II.

305. In like manner, if we have the amplitude, OS, and latitude of a place, given, we shall easily find the sun's declination corresponding to this amplitude; that is, $\textit{sin. } DS = \frac{\textit{sin. } OS \times \textit{sin. } DOS}{R}$.

LOGA-

LOGARITHMICALLY.

9.683229	= *log. sin.*	28° 49′ 45″.
9.818247	= *log. sin.*	41° 9′ 0″.
9.501476	= *log. sin.*	18° 30′ 0″.

PROBLEM V.

306. *Given the latitude of a place, and the sun's place in the ecliptic; to find the point of the equator which comes to the horizon at the same time with the sun, or* (which is the same) *the oblique ascension corresponding to the said latitude and place of the sun.*

SOLUTION.

Let us suppose, as in the first Problem, that the sun's place in the ecliptic is 18° 24′ of Taurus: then (by the same Problem) his declination will be 17° 19′ 49″; consequently, in the right-angled triangle DOS, we shall know the leg DS with the angle DOS = the complement of the latitude, in order to find the leg OD. To do which, the general Table for right-angled triangles gives, $sin.\ OD = \frac{tang.\ DS \times R}{tang.\ DOS}$.

LOGARITHMICALLY.

9.494217	= *log. tang.* DS	= 17° 19′ 49″.
0.058541	= *arith. comp. log. tang.*	41° 9′ 0″.
9.552758	= *log. sin.* OD	= 20° 55′ 15″.

Then, if from the right ascension, DE, found in *art.* 302 — — — —	45° 55′ 58″
we subtract the arc thus found — — —	20° 55′ 15″
we shall have for the oblique ascension, EO	25° 0′ 43″

SCHOLIUM.

307. It is well known that at the equator the days and nights are constantly equal, because in the right sphere the points D and O coincide. In an oblique sphere, the arc DO expresses the quantity which must be added to, or subtracted from, 90°, in

in order to obtain half the length of the day at a place whoſe latitude is northerly, according as the ſun's declination is towards the north or ſouth. ——It is manifeſt that, we have paid no regard to the augmentation cauſed by refraction.

Problem VI.

308. *Given the latitude, and ſun's place in the ecliptic; to find the angle made by the ecliptic with the horizon at the inſtant of ſun-riſe.*

SOLUTION.

Let the ſun's place be ſtill ſuppoſed to be $18°$ $24'$ of Taurus; the latitude $48° 51'$, and conſequently its complement AOH or DOS $41° 9'$: then (as already found in *art.* 299) the arc DS will be $17° 19' 49''$; whence, by making the calculation, as in *Prob.* IV. *ſin.* OS, the ortive amplitude will be found $= 9.655800$; alſo EO (obtained from the ſame *data* in the laſt Problem) $= 25°$ $0' 43''$; and therefore, by the common analogy between the *ſines* of ſides and thoſe of their oppoſite angles, we ſhall get, $\textit{ſin.}\ \mathrm{ESO} = \frac{\textit{ſin.}\ \mathrm{SEO} \times \textit{ſin.}\ \mathrm{EO}}{\textit{ſin.}\ \mathrm{OS}}$.

LOGARITHMICALLY.

$9.600263 = \textit{log. ſin.}\ \mathrm{SEO} = 23° 28' 30''$.
$9.626141 = \textit{log. ſin.}\ \mathrm{EO} = 25° 0' 43''$.
$0.344200 = \textit{arith. comp. log. ſin.}\ \mathrm{OS}$.
$9.570604 = \textit{log. ſin.}\ \mathrm{ESO} = 21° 50' 32''$.

SCHOLIUM.

309. By calculations, nearly ſimilar to thoſe made in the two preceding Problems, it would be eaſy to find the arc IE, of the ecliptic, contained between the meridian and interſection of the ecliptic and equator. For, having obtained the oblique aſcenſion, EO, the arc AE, of the equator which is the complement thereof, will become known; whence, in the right-angled triangle IAE, the ſide

side AE with the angle E equal to the obliquity of the ecliptic will be given.——We might likewise find the arc of the meridian intercepted between the equator and ecliptic; as well as the angle made by the ecliptic with the meridian, and the arc, IS, comprised between the meridian and horizon.

We shall now subjoin a Problem whose solution depends only upon a common right-angled spherical triangle, though Mr. *Maupertuis* hath for that end made use of the differential (fluxional) calculation.

PROBLEM VII.

310. *The declination of a star, and the latitude of a place being given; it is required to find,* 1°, *the moment when this star appears to ascend or descend perpendicularly to the horizon;* 2°, *the altitude of the star in a vertical; and* 3°, *the angle which the said vertical makes with the meridian.*

SOLUTION.

With a very little attention to the nature of the Problem it will appear, that the star cannot possibly ascend or descend perpendicularly to the horizon, unless the parallel, which it describes, can be supposed to have some vertical touching it; and that the star will be found in the point of contact of the said parallel and vertical; yet that this thing can only prevail twice in a revolution of 24 hours. Moreover, it is easy to perceive, that no parallel can have any vertical to touch it, except its distance from the equator be greater than the latitude of the place; for provided it be less than, and on the same side of the equator with, the given latitude, it is very obvious that, in this case, all verticals whatever will cut it. This being premised, let ZST be a vertical touching the

parallel

parallel GSM at the point S, the ſuppoſed place of a ſtar, whoſe declination hath for its complement the arc PS: then, on account of the point of contact of this vertical and parallel, it is manifeſt that, the arc PS is the leaſt that can be drawn from the point P to the vertical ZST: conſequently, the triangle PSZ will be right-angled at S, and we ſhall know therein (beſide the right angle) the ſide PS = the complement of the ſtar's declination, and the ſide PZ = the complement of the latitude: whence, if the difference between the right aſcenſion of the ſun and ſtar be known, the hour-angle ZPS will give the inſtant when this ſtar appears to aſcend perpendicularly to the horizon; whilſt the ſide ZS will give the complement of its altitude, and the angle PZS the poſition of the vertical required with reſpect to the meridian.——If then L be put for the latitude of the place; D for the declination of the ſtar; H′ for its altitude above the horizon, when it appears to aſcend vertically; H for the hour-angle ZPS, and V for the angle made by the vertical and meridian; the formulæ for right-angled triangles will give; $cof.\ H = \frac{cot.\ D \times R}{cot.\ L}$; $fin.\ H' = \frac{cof.\ L \times R}{cof.\ D}$, and, $fin.\ V = \frac{cof.\ D \times R}{cof.\ L}$. ——The applications to the Logarithms will be ſufficiently eaſy without examples.

PROBLEM VIII.

311. *Given the latitude of a place, and the ſun's declination; to find the time of his riſing and ſetting.*

SOLUTION.

Suppoſing the declination 18° 30′ northerly, and the latitude 48° 51′; it is manifeſt that, in the right-angled triangle PSR, there will be given PR equal to the latitude, and PS the complement

ment of the declination; and therefore for the angle RPS (the thing required) we ſhall have, *coſ.* $RPS = \frac{tang.\ PR \times R}{tang.\ PS}$.

LOGARITHMICALLY.

10.058541 = *log. tang.* PR		= 48° 51′ 0″.
9.524520 = *arith. comp. log. tang.*		71° 30′ 0″.
9.583061 = *log. coſ.* RPS		= 67° 29′ 17″.

Now, in order to know what hour of the morning this value of RPS correſponds to, we muſt have recourſe to the annexed Table; and we ſhall readily find, that

For	15°	reckon	1 *hour.*	on the day when the declination is 18° 30′ northerly, at a place whoſe latitude is 48° 51′, the ſun riſes 29 minutes 57 ſeconds after 4 o'clock.
	1°		4 *minutes.*	
	15′		1 *minute.*	
	1′		4 *ſeconds.*	
	15″		1 *ſecond.*	
	1″		4 *thirds.*	
	15‴		1 *third.*	

SCHOLIUM.

312. If the amplitude was known inſtead of the latitude, the angle RPS would be obtained by this proportion; *As the coſine of the declination is to the ſine total, ſo is the coſine of the amplitude to the ſine of the hour-angle,* RPS: and contrarily, if the time of ſun-riſe or ſun-ſet was given, the ortive or occaſive amplitude would be found by ſaying; *As the ſine total is to the coſine of the declination, ſo is the ſine of the hour-angle to the coſine of the amplitude.* ——We may likewiſe perceive that, if the ſun's declination and time of his riſing were given, the latitude of the place would be eaſily found; ſince, in the right-angled triangle SPR, we ſhould know the hypothenuſe and angle P.——We might alſo find the ſun's declination from the hour of his riſing and the latitude.——Moreover, this ſolution may be applied to the riſing or ſetting of a

planet or ſtar, provided we know its declination and right aſcenſion: for, if the ſun's right aſcenſion be found for the ſame day by means of his declination, the difference between it and that of the ſtar reduced into hours, minutes and ſeconds, by the foregoing Table, will give the time that the ſtar riſes before, or after, the ſun: by which means we ſhall be able to know at what time of the year a ſtar can be ſeen or not ſeen.——We may alſo make uſe of this Problem in order to find the ſun's horizontal refraction: for, ſuppoſing the true, and alſo apparent, time of his riſing to be known, if we take their difference, and reduce it into parts of the equator; then calculate the ſide ZS, and take 90° therefrom, the remainder will ſhew the quantity of his elevation occaſioned by refraction.

PROBLEM IX.

313. *Given the latitude of a place; the ſun's declination, and his altitude; to find the hour of the day.*

SOLUTION.

Fig. 35. Let the latitude be 48° 51′; the ſun's declination 18° 30′ northerly, and his altitude 52° 35′: then in the triangle PZS we ſhall know the three ſides, in order to find the angle ZPS: which will be eaſily done by the formulæ (*art.* 166), $\text{ſin.}\ \frac{1}{2}P = \frac{\sqrt{\text{ſin.}\ \frac{1}{2}s-b \times \text{ſin.}\ \frac{1}{2}s-c}}{\sqrt{\text{ſin.}\ b \times \text{ſin.}\ c}}$; wherein s denotes the ſum of the three ſides, and, b, c, the ſides adjacent to the angle required.——See the *Logarithmical* operation.

$37^\circ\ 25'$	$75^\circ\ 2'$	
$71^\circ\ 30'$	$41^\circ\ 9'$	
$41^\circ\ 9'$	$33^\circ\ 53' = \frac{1}{2}\,s - b.$	
$150^\circ\ 4' = s$	$75^\circ\ 2'$	
$75^\circ\ 2' = \frac{1}{2}\,s$	$71^\circ\ 30'$	
	$3^\circ\ 32' = \frac{1}{2}\,s - c.$	

9.746247 = *log. sin.* — — $33^\circ\ 53'$.
8.789787 = *log. sin.* — — $3^\circ\ 32'$.
0.023043 = *arith comp. log. sin.* $71^\circ\ 30'$.
0.181753 = *arith. comp. log. sin.* $41^\circ\ 9'$.

18.740830

9.370415 = *log. sin.* $\frac{1}{2}$ P $= 13^\circ\ 34'\ 14'$ ∴ P $= 27^\circ\ 8'\ 28''$; the time corresponding to which is 10 hours 11 minutes 26 seconds and 8 thirds of the morning, if the angle ZPS be to the east of the meridian; or 48 minutes 33 seconds and 52 thirds after 1 in the afternoon, if it be to the west.

SCHOLIUM.

314. This Problem points out to us an easy method of finding the duration of twilight. For, if we conceive the sun to be depressed 18° below the horizon (as the crepuscular circle is usually supposed to be), the arc ZS will be 108°: then if we calculate (as in the present Problem) the hour-angle ZPS, and afterwards subtract it from 180°, the remainder, reduced into parts of time, will give the beginning of twilight, or break of day. If it is the end of twilight which we would find, the angle ZPS itself must be reduced into parts of time.

OBSERVATION.

In the performance of this Problem, we must take care to correct the sun's altitude by refraction and parallax (if material).——Tables of these

corrections may be met with in moſt Treatiſes of Aſtronomy.

Problem. X.

315. *Given the latitude of a place, with the ſun's declination and altitude; to find the angle that the vertical, on which the ſun is, makes with the meridian.*

SOLUTION.

From the *data* it is manifeſt that, in the triangle PZS the three ſides will be known, in order to find the angle PZS. Now this may be obtained by the method given in the laſt Problem; and therefore, if the parts given be ſuppoſed to have the ſame value in this that they had in that, the *Logarithmical* operation will ſtand thus:

$33^\circ\ 53' = \frac{1}{2}s-b.$	9.746248	$=$ *log. ſin.* $\frac{1}{2}s-b.$
$37^\circ\ 37' = \frac{1}{2}s-c.$	9.785597	$=$ *log. ſin.* $\frac{1}{2}s-c.$
	0.181753	$=$ *arith. comp. log. ſin.* $41^\circ\ 9'.$
	0.216377	$=$ *arith. comp. log. ſin.* $35^\circ\ 25'.$
	19.929975	
	9.964987	$=$ *log. ſin.* $\frac{1}{2}$PZS $= 67^\circ\ 18'\ 3''$:

whence PZS $= 134^\circ\ 36'\ 6''$; and conſequently, the angle that the vertical, on which the ſun is, makes with the meridian $45^\circ\ 23'\ 54''$.

SCHOLIUM.

316. From this operation it will be eaſy to find the four cardinal points: ſince, if with a line, repreſenting the vertical on which the ſun is, an angle be made equal to that above found, the meridian will be obtained. Now this method, provided we take care to correct the ſun's altitude by refraction, will be found very commodious for determining the meridian, as the changes in declination which prevail in correſponding altitudes are

hereby

hereby entirely avoided. It ſerves likewiſe to determine the variation of the compaſs.

PROBLEM XI.

317. *Given the latitude; ſun's declination, and hour of the day; to find the angle that the vertical, on which the ſun then is, makes with the meridian.*

SOLUTION.

Here it is manifeſt that, in the ſpherical triangle PZS the two ſides, PZ, PS, with their included angle, ZPS, are known. Now let us ſuppoſe the declination to be 18° 30′, and that the ſun was on the vertical, whoſe poſition is required, at 11 minutes 26 ſeconds paſt 10: then, by the Table given in *Prob.* VIII, the hour-angle correſponding to this time will be 27° 8′ 30″; the arc PS will be ſtill 71° 30′, and PZ 41° 9′. This being premiſed, in order to find the angle PZS, let fall the arc SR perpendicularly upon the ſide PZ from the vertex of the (third) angle S (as required in *Table* II. for the reſolution of oblique-angled ſpherical triangles), and we ſhall in the firſt place get, for the part there called I. *ſeg. tang.* PR $= \frac{\textit{coſ.}\ ZPS \times \textit{tang.}\ PS}{R}$: or,

LOGARITHMICALLY,

9.949332	= *log. coſ.*	27° 8′ 30″.
10.475480	= *log. tang.*	71° 30′ 0″.
10.424812	= *log. tang.*	69° 23′ 37″ = PR;

then the ſecond ſegment, RZ, will be 28° 14′ 37″; and therefore, by means of the ſame Table, we ſhall laſtly have, *tang.* PZS $= \frac{\textit{tang.}\ ZPS \times \textit{ſin.}\ PR}{\textit{ſin.}\ RZ.} =$ 45° 23′ 53″.—— See the *Logarithmical* operation.

9.705815

$9.709815 = log.\ tang.\ ZPS = 27^\circ\ 8'\ 30''.$
$9.971285 = log.\ sin.\ PR = 69^\circ\ 23'\ 37''.$
$0.324936 = arith.\ comp.\ log.\ sin.\ 28^\circ\ 14'\ 37''.$
$10.006036 = log.\ tang.\ PZS = 45^\circ\ 23'\ 53''$ or $134^\circ\ 36'\ 7''.$

SCHOLIUM.

318. This Problem may ſerve for finding the declination of any vertical plane. We may likewiſe apply it in order to determine the poſition of an avenue or well-built ſtreet, by obſerving the moment when the ſun's ſhadow paſſes by the bottom of the trees or walls thereof. —— Moreover, if the ſun's altitude above the horizon be likewiſe known at the given time, the preceding calculation will be rendered conſiderably more ſimple, and reduced to this analogy; *As the coſine of the ſun's altitude is to the ſine of the hour-angle, ſo is the coſine of the declination to the ſine of the angle made by the vertical and meridian.*

PROBLEM XII.

319. *Given the hour of the day, with the ſun's declination and altitude; to find the latitude of the place.*

SOLUTION.

Fig. 35. In this Problem, it is manifeſt that, the ſide PS = the complement of the ſun's declination; the ſide ZS = the complement of his altitude, and the angle P, are given, to find the ſide ZP. Now let us ſuppoſe, as in the preceding Problems, that the ſun's declination is $18^\circ\ 30'$; that his altitude is $52^\circ\ 35'$, and conſequently the arc ZS = $37^\circ\ 25'$; as alſo that the time is 11 minutes 26 ſeconds paſt 10, and of courſe the angle P = $27^\circ\ 8'\ 30''$: then, letting fall the perpendicular SR from the angle S, PR will, in the firſt place be determined by the formula, in the general Table for oblique-angled triangles, *tang.* I. *ſeg.* = *coſ. giv. ang.* × *tang. of its adj. ſide*: in the next place, we ſhall have, *coſ.*

$cos.$ II. $seg. = cos.$ RZ $= \frac{cos. \text{I. } seg. \times sin. \, 52^\circ \, 35'}{sin. \, 18^\circ \, 30'}$; from whence the value of the side PZ = the complement of the latitude will be immediately deduced.——See the operation at length.

$$\begin{array}{ll} 1^\circ. & 9.949332 = log.\, cos. \;\; 27^\circ \;\; 8' \; 30''. \\ & \underline{10.475480} = log.\, tang.\; 71^\circ \; 30' \;\; 0''. \\ & \underline{10.424812} = log.\, tang.\; 69^\circ \; 23' \; 37'' = \text{PR}. \end{array}$$

$$\begin{array}{ll} 2^\circ. & 9.546475 = log.\, cos. \; \text{- - - - -} \; 69^\circ \; 23' \; 37''. \\ & 9.899951 = log.\, sin. \; \text{- - - -} \; 52^\circ \; 35' \;\; 0''. \\ & \underline{0.498524} = arith.\, comp.\, log.\, sin.\; 18^\circ \; 30' \;\; 0''. \\ & 9.944950 = log.\; cos.\; \text{RZ} = 28^\circ \; 14' \; 37'': \end{array}$$

then, subtracting the second segment = 28° 14′ 37″ from the first = 69° 23′ 37″, we shall get 41° 9′ for the complement of the latitude; and consequently, 48° 51′ for the latitude itself.

PROBLEM XIII.

320. *Given the sun's declination; his altitude, and the azimuth, or angle made with the meridian by the vertical on which the sun is found at the moment of observation; to determine the latitude of the place.*

SOLUTION.

It is evident that, in the triangle PZS the sides, PS, ZS, with the angle PZS opposite to the side PS, are given: therefore, in order to obtain the side PZ, we must (according to the Table for oblique angled spherical triangles) let fall from the vertex of the angle S an arc, SR, perpendicularly upon the unknown side; and we shall get; 1°, $tang.\,\text{RZ} = \frac{cos.\,\text{PZS} \times tang.\,\text{ZS}}{\text{R}}$, and 2°, $cos.\,\text{PR} = \frac{cos.\,\text{RZ} \times cos.\,\text{PS}}{cos.\,\text{Z}}$: the values whereof, supposing the sun's declination to be 18° 30′, his altitude above the horizon 52° 35′, and the angle, PZS, made by the vertical with the meridian 134° 36′ 7″, will be obtained *Logarithmically*, as follows:

1°.

$1°.\ 9.846448 = log.\ cos.\ 134°\ 36'\ 7''.$
$\underline{9.883672} = log.\ tang.\ 37°\ 25'\ 0''.$
$\underline{9.730120} = log.\ tang.\ 28°\ 14'\ 37'' = RZ.$

$2°.\ 9.944948 = log.\ cos.\ \text{- - - - -}\ 28°\ 14'\ 37''.$
$9.501476 = log.\ sin.\ \text{- - - - -}\ 18°\ 30'\ 0''.$
$\underline{0.100049} = arith.\ comp.\ log.\ sin.\ 52°\ 35'\ 0''.$
$9.546473 = log.\ cos.\ RP = 69°\ 23'\ 37''.$

then, subtracting 28° 14′ 37″ from 69° 23′ 37″, we shall find the side PZ = the complement of the latitude to be 41° 9′; whence the latitude itself is 48° 51′.

SCHOLIUM.

321. From what hath been said it may be readily collected, that if any three of these six things, the latitude, the sun's altitude, his declination, the hour-angle, azimuth and angle at the sun, be given at pleasure, the other three will necessarily become known by the rules of Trigonometry: for which reason we hope to be excused for not particularizing all the different cases that result from their combinations.

PROBLEM XIV.

322. *The longitude and latitude of a star being given; to find its declination.*

SOLUTION.

Suppose the longitude of the star to be 198° 27′, and its latitude 31° 2′ northerly: then, it is evident that, in the triangle PKS, we know the side PK = to the obliquity of the ecliptic = 23° 28′ 30″; the side KS equal to the complement of the latitude = 58° 58′, and the angle PKS = to the longitude of the star less 90°, in order to find the side PS, the complement of the declination required. Now to do this, let fall (as the general Table for oblique-angled triangles requires) from one of the unknown angles, S, a perpendicular arc, SR, upon the

he opposite side, PK (which will fall without the angle S, because the angle PKS is obtuse, and give the arc KR for the part denoted by *seg.* I. and PK + KR or PR for *seg.* II.), so will; 1°, *tang.* KR $= \frac{cos.\,PKS \times tang.\,KS}{R}$, and 2°, *cos.* PS or *sin.* DS $= \frac{cos.\,KS \times cos.\,PR}{cos.\,KR}$.——See the *Logarithmical* operation.

1°. 9.500342 = *log. cos.* PKS = 108° 27′ 0″.
10.220654 = *log. tang.* KS = 58° 58′ 0″.
9.720996 = *log. tang.* KR = 27° 44′ 41″.
∴ PR = 51° 13′ 11″.

2°. 9.712260 = *log. cos.* KS.
9.796804 = *log. cos.* PR.
0.054042 = *arith. comp. log. cos.* KR.
9.563106 = *log. sin.* 21° 26′ 58″ = DS = *dec. required.*

PROBLEM XV.

323. *Supposing the same things given as in the preceding Problem; it is required to find the right ascension of the same star, as also the angle made by a meridian and the circle of longitude passing through this star.*

SOLUTION.

From the *data* it will be no difficult matter to perceive, that in the triangle PKS two sides and their included angle are still known; and likewise that the star's right-ascension will be determined by subtracting the angle KPS from 270°. Now, in order to obtain the value of this angle, together with that of the angle at the star at the same time, it will be necessary to make use of some of the analogies of *Neper*; and those that appear to conduce to this end will be found by *Tab.* III. to be the two following: *Fig.* 36.

1°, *As the sine of half the sum of the sides,* PK *and* KS, *is to that of half their difference, so is the cotangent of half their included angle,* PKS, *to the tangent of half the difference of the angles above the base*; and 2°, *as the cosine of half the sum of these sides is to the cosine of half their difference, so is the cotangent of half*

their included angle to the tangent of half the ſum of the angles above the baſe.——Here follows the *Logarithmical* operation:

$$58^\circ\ 58'\ 0'' = KS. \qquad 58^\circ\ 58'\ 0''.$$
$$23^\circ\ 28'\ 30'' = PK. \qquad 23^\circ\ 28'\ 30''.$$
$$82^\circ\ 26'\ 30''. \qquad 35^\circ\ 29'\ 30''.$$
$$41^\circ\ 13'\ 15'' = \frac{KS + PK}{2} \qquad 17^\circ\ 44'\ 45'' = \frac{KS - PK}{2}$$
$$108^\circ\ 27'\ 0'' = PKS.$$
$$54^\circ\ 13'\ 30'' = \frac{PKS}{2}, \textit{ whoſe. comp.} = 35^\circ\ 46'\ 30''.$$

9.484008 = *log. ſin.* 17°44′45″.	9.978827 = *log. coſ.* 17°44′45″.
9.857670 = *log. tang.* 35°46′30″.	9.857670 = *log. tan.* 35°46′30″.
0.181139 = *a.c.l. ſin.* 41°13′15″.	0.123682 = *a.c.l. coſ.* 41°13′15″.
9.522817 = *log. tang.* 18°25′56″.	9.960179 = *log. tan.* 42°22′36″:

then, as half the ſum of the angles at P and S is 42° 22′ 36″, and half their difference 18° 25′ 56″, the angle KPS will be 60° 48′ 32″, and of courſe the ſtar's right aſcenſion 209° 11′ 28″; and the angle at the ſtar 23° 56′ 40″.

SCHOLIUM.

324. The right aſcenſions and declinations of the ſtars, as likewiſe their longitudes and latitudes, are of exceeding great and frequent uſe in Aſtronomy. By their means the reſpective poſitions and ſituations of the ſtars are aſcertained and determined, and from thence ſuch catalogues thereof formed and deduced, as are neceſſary for comparing the motions of the Planets, as well as of the Comets that may at different times be obſerved to appear. However, the preference is uſually given to the longitudes and latitudes; becauſe, it hath been for a long time believed that theſe elements undergo no change, particularly, if care be taken to reckon the longitudes from ſome *fixed* ſtar; whilſt the preceſſion of the equinoxes occaſions a conſiderable change in the right aſcenſions and declinations. But abating all conſiderations of this nature, it is eaſy to perceive that, the ſeveral elements of the Theory of the ſtars may be deduced from one another,

other, by a very great variety of combinations; to particularize which in this place would be foreign to our design. The right ascensions and declinations of the stars may be used, as well as those of the sun, to determine the latitudes of places on the earth: from thence their ortive or occasive amplitudes may be deduced, and of consequence those of such stars as may have the same positions in the sphere: from thence it may be found, what stars never set in a given latitude; likewise, by their means it may be known, how long a star transits the meridian before, or after, the sun; and of consequence when the heliacal rising or setting of a star commences, that is, at what time of the year the sun's rays render it visible or invisible. The most simple methods, that have been adopted for determining the positions of the stars, consist in observing their meridian altitudes and the instant of their transiting the meridian, as by that means their right ascensions and declinations are immediately obtained; and from thence their longitudes and latitudes, by the converse of the two Problems last solved.

PROBLEM XVI.

325. *Given the sun's place in the ecliptic; the hour of the day or night, and the latitude of the place; also the altitude of a star, and the angle which the vertical, on which it is observed, makes with the meridian; to determine the declination of this star, and its right ascension.*

SOLUTION.

Let us suppose the sun's place to be 18° 24′ of Taurus; the time 20 minutes past 9 in the evening, and the latitude of the place 48° 51′; also the altitude of the star, in the vertical circle on which it is observed, to be 43° 15′, and the azimuth 47° 24′ from the south: then, it is manifest that, in the triangle PZS, we shall know the two sides, PZ, ZS, with their included angle, to find the arc *Fig.* 35.

DS which repreſents the declination of the ſtar, and the diſtance of the point D from the firſt point of Aries. Now, to obtain the declination (which will be done by means of the ſide PS), let fall from the vertex of the angle S a perpendicular arc, SR, upon the ſide PZ, and the Table for oblique-angled triangles will give theſe two formulæ; 1°, *tang.* RZ = *coſ.* PZS × *tang.* ZS, and 2°, *coſ.* PS or *ſin.* DS = $\frac{\textit{coſ.}\,ZS \times \textit{coſ.}\,PR}{\textit{coſ.}\,RZ}$: whoſe values will be found *Logarithmically*, as follows:

9.830509 = *log. coſ.* PZS.
10.026546 = *log. tang.* ZS.
9.857055 = *log. tang.* RZ = 35° 44′ 11″

9.835807 = *log. coſ.* ZS.
9.593900 = *log. coſ.* PR.
0.000598 = *ar. comp. log. coſ.* RZ.
9.520305 = *log. ſin.* 19° 21′ 5″ = DS.

The ſtar's declination being thus diſcovered to be equal 19° 21′ 5″, the next thing to be done is to find its right aſcenſion. In order to which, we muſt in the firſt place determine the hour-angle ZPS by the common analogy between the *ſines* of ſides and thoſe of their oppoſite angles, which gives, *ſin.* ZPS = $\frac{\textit{ſin.}\,PZS \times \textit{ſin.}\,ZS}{\textit{ſin.}\,PS}$ = *ſin.* 34° 37′ 43″, as the ſubſequent operation will evince;

9.866935	= *log. ſin.* PZS	=	132° 36′ 0″.
9.862353	= *log. ſin.* ZS	=	46° 45′ 0″.
0.025257	= *arith. comp. log. ſin.*		70° 38′ 55″.
9.754545	= *log. ſin.* ZPS	=	34° 37′ 43″:

then adding this angle, the ſun's diſtance from the meridian = 140° (found by reducing the given time 9 hours 20 minutes into parts of the equator), and his right aſcenſion = 45° 55′ 58″ (*art.* 302), into one ſum, we ſhall get 220° 33′ 41″ for the ſtar's right aſcenſion ſought.——If we would know the time of the ſtar's paſſing over the meridian, we need only reduce 34° 37′ 43″ into parts of time, add the reſult, 2 hours 18 minutes 31 ſeconds, to 9 hours 20 minutes, and we ſhall find 11 hours 38 minutes 31 ſeconds in the evening for the time required.

PRO-

PROBLEM XVII.

326. *The right ascensions and declinations of two fixed stars, observed on the same vertical given in position, together with their distance or the arc of this vertical contained between them, being known; it is required to determine the latitude of the place of observation.*

SOLUTION.

From the *data* it will manifestly appear, that the three sides of the triangle PS*s* are given, as well as the angle at P; since the arcs, PS, P*s*, are the complements of the declinations of the two stars (known by the hypothesis); S*s* their distance on the vertical given in position, and the angle SP*s* the measure of the difference of their right ascensions: wherefore, we shall have, *sin.* S*s* : *sin.* SP*s* : : *sin.* *s*P : *sin.* PSZ $= \frac{\textit{sin. } s\text{P} \times \textit{sin. } \text{SP}s}{\textit{sin. } \text{S}s}$. By this means two angles, and a side opposite to one of them, will be actually known in the triangle PZS, viz. the azimuth or angle PZS; the angle PSZ, and the side PS: whence we shall have, *sin.* PZS : *sin.* PSZ : : *sin.* PS : *sin.* PZ, or, by substituting for *sin.* PSZ its preceding value and taking away the fraction, *sin.*PZS × *sin.*S*s* : *sin.* *s*P × *sin.*SP*s* : : *sin.*PS : *sin.*PZ. Fig. 37.

Now, in order to apply this solution to a particular example, suppose the distance S*s* to be $= 28° \ 30'$; the right ascension of S $= 78° \ 24'$, and its declination $= 27° \ 25'$; the right ascension of $s = 104° \ 52'$, and its declination $= 12° \ 18'$; and lastly the azimuth AZK $= 73° \ 36'$: then will the angle SP*s* be $= 26° \ 28'$; the arc PS $= 62° \ 35'$, and the arc P*s* $= 77° \ 42'$. This being premised, the value of PZ will be easily obtained, *Logarithmically*, as follows:

9.948257 =	*log. sin.* - - - - -	62° 35′ 0″	= PS.
9.989915 =	*log. sin.* - - - - -	77° 42′ 0″	= P*s*.
9.649020 =	*log. sin.* - - - - -	26° 28′ 0″	= SP*s*.
0.018039 =	*arith. comp. log. sin.*	73° 36′ 0″	= PZS.
0.321337 =	*arith. comp. log. sin.*	28° 30′ 0″	= S*s*.
9.926568 =	*log. sin.* - - - - -	57° 36′ 42″	= *comp. of lat.*

327.

COROLLARY I.

327. It is very evident, that the hour-angle ZPS may be alſo found; and therefore, if the ſun's place and his right aſcenſion be known, the difference between it and that of the ſtar, S, will become known: then, if from this difference the arc AD be taken, the remainder will give the arc of the equator contained betwixt the meridian of the place, and that paſſing through the ſun; from whence the moment of obſervation will be eaſily deduced.——It is no leſs evident, that we may likewiſe find, by purſuing the calculation, the altitude of each of the ſtars at the inſtant of obſervation.

COROLLARY II.

328. If, inſtead of the azimuth of the two ſtars, the altitude of one of them was given, it will be readily perceived that, by a calculation nearly ſimilar, the elevation of the pole might be obtained: for we ſhould ſtill know the three ſides of the triangle PS*s* with the angle P, and thence the angle PSZ contained between the given ſides, PS, ZS, of the triangle, PSZ.

COROLLARY III.

329. If, beſides the declinations and right aſcenſions of the ſtars obſerved on the ſame vertical, the hour-angle ZPS was given, we ſhould again be able to obtain the latitude of the place; ſince, in the triangle PSZ, we ſhould then have the angles, ZPS, ZSP, above the ſide SP, known by ſuppoſition.

COROLLARY IV.

330. Laſtly, if the diſtance and declinations of the ſtars, together with the altitude of one of them, were given, every thing beſides might be found as in the Problem itſelf. And hence we may perceive, how uſeful a thing it is to be furniſhed with accurate Tables of the declinations, right aſcenſions,

ſions, diſtances, longitudes and latitudes, of the ſtars, as from thence the latitudes of places may be found with the greateſt imaginable facility; nay, even without the neceſſity of uſing any large inſtruments, a plumb-line being ſufficient for obſerving whether two ſtars be ſituate on the ſame vertical or not.

PROBLEM XVIII.

331. *Given the longitudes and latitudes of two places upon the terreſtrial globe; to find their itinerary diſtance, or* (which is the ſame) *the arc of the great circle contained between them.*

SOLUTION.

Let the points, S, *s*, denote the propoſed places: then, it is manifeſt that, in the triangle SP*s* we ſhall know the two ſides, SP, *s*P, the complements of their latitudes, together with the angle P = the difference of their longitudes; whence the ſide S*s* will be found by the third caſe of the Table for oblique-angled ſpherical triangles; after which we need only reduce the reſulting number of degrees, minutes and ſeconds, into leagues, by reckoning 25 to a degree, and the thing required will be effected. *Fig.* 27.

PROBLEM XIX.

332. *Given the latitudes and diſtance of two cities; to find their difference in longitude.*

SOLUTION.

The firſt thing to be done, in this Problem, is to reduce the given diſtance into degrees, minutes and ſeconds, of a great circle: then we ſhall have a ſpherical triangle, SP*s*, whereof the three ſides will be known; from whence the angle P will be eaſily found, which is the meaſure of the difference in longitude ſought.

COROLLARY.

333. Hence, if the poſition of one of the two Cities be determined, that of the other will be determined

termined likewiſe; and with ſo much the greater exactneſs, as the given diſtance ſhall be meaſured or given with the greater preciſion. And contrarily, if the difference of the longitudes and the diſtance of two Cities be given, together with the latitude of one of them, the latitude of the other City will be alſo had: but we ſeldom make uſe of this method, becauſe there is an infinite variety of better methods for finding the latitude, as we may be readily convinced from what hath been previouſly ſaid.

SCHOLIUM.

334. The preceding Problems, containing nearly all the caſes of right or oblique-angled ſpherical triangles, are ſufficient for pointing out the manner of obtaining the *Numerical* ſolutions of the different Problems of Trigonometry, by means of the formulæ demonſtrated in the foregoing Chapters. In this Chapter we had nothing farther in view, than the application of a few pertinent examples to the *Logarithms*; for which reaſon we have omitted taking notice of the different modifications whereof the things given might be ſuſceptible, with reſpect to parallax, refraction, the aberration of light, or nutation of the earth's axis. Theſe Theories can, properly ſpeaking, belong only to a complete Treatiſe of Aſtronomy: and therefore, we flatter ourſelves that it will be deemed enough for us to have laid down methods in this Work of ſolving the ſeveral trigonometrical caſes, that may chance to occur in the ſtudy of this ſcience; the difficulties whereof will riſe in proportion to that rigor and exactneſs, wherewith we may be deſirous of calculating the various *phænomena* therein conſidered.

FINIS.

Fig. 1.

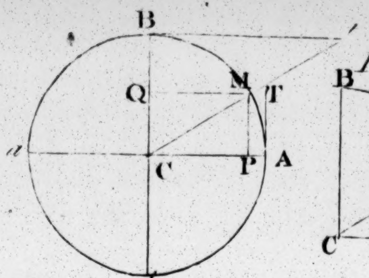
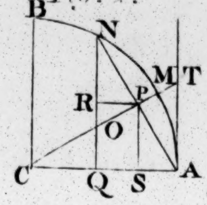


Fig. 2.



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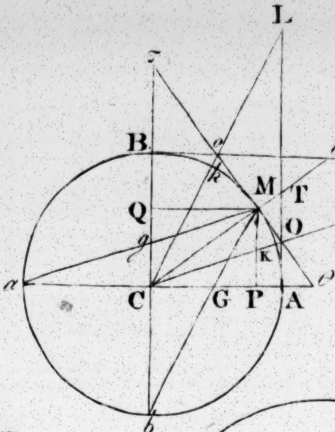

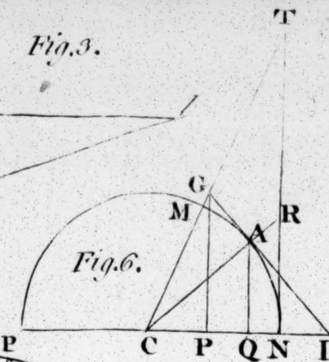

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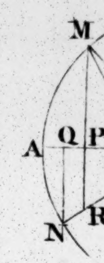

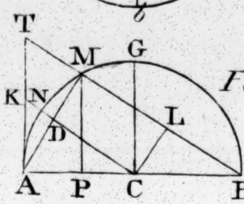

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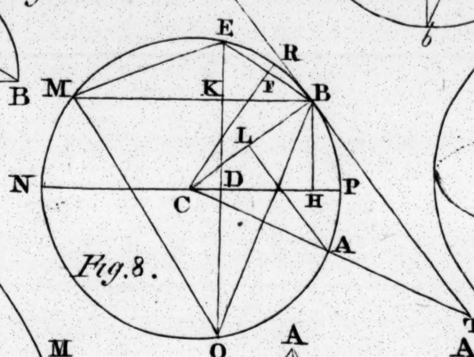

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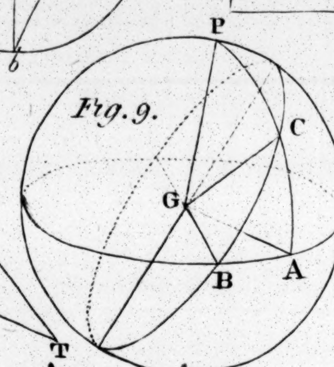

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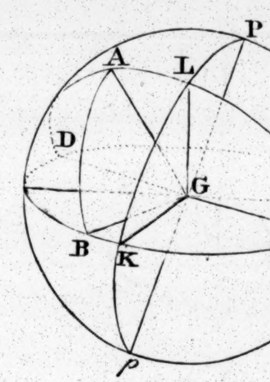

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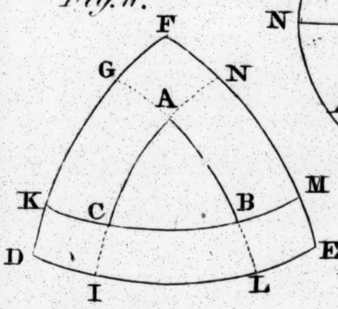

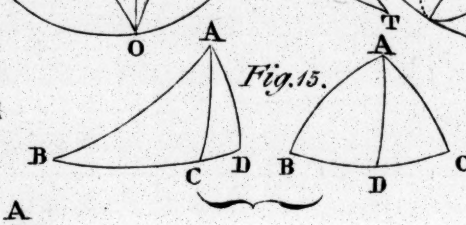

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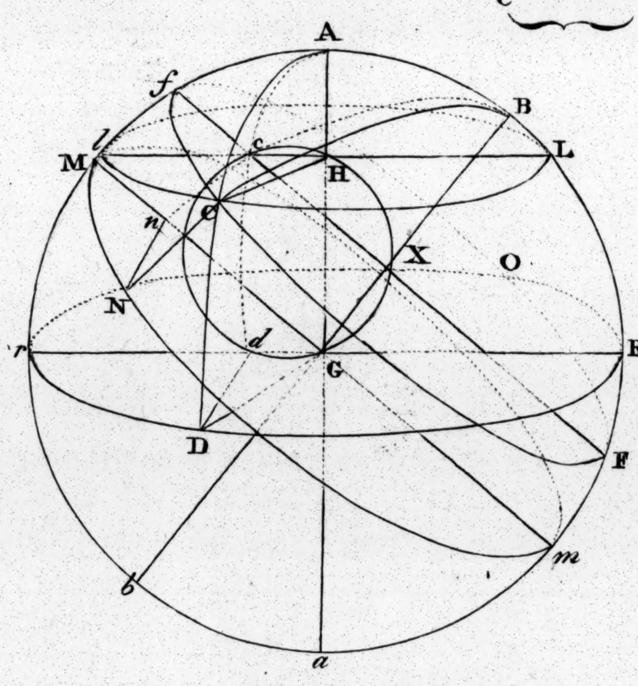

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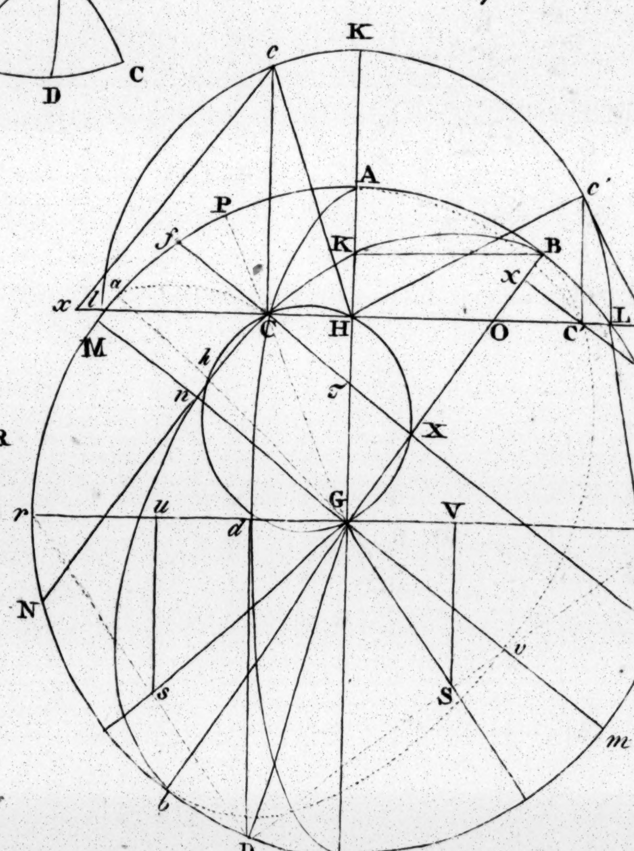
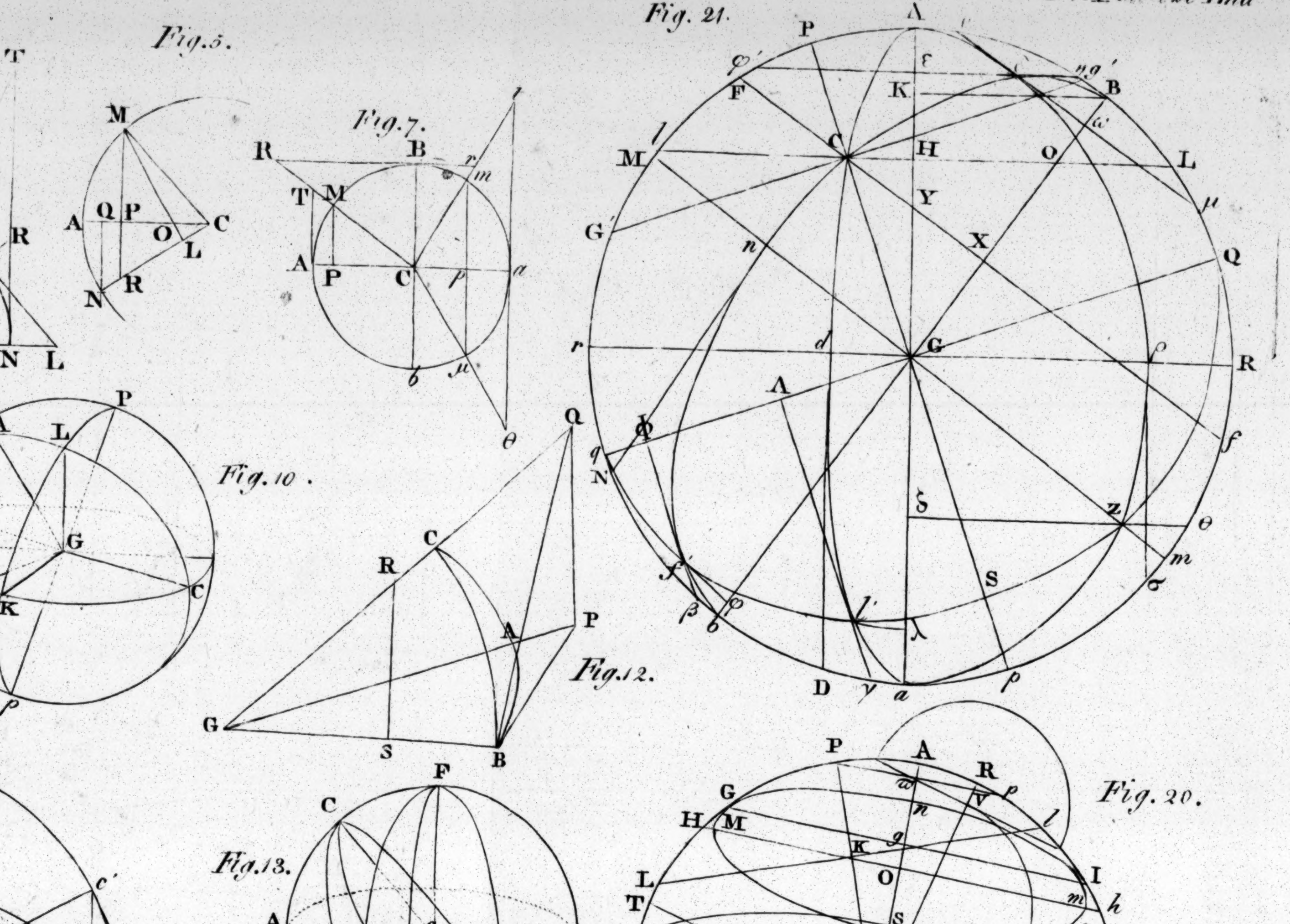


Fig. 17.

Pl. I. at the End



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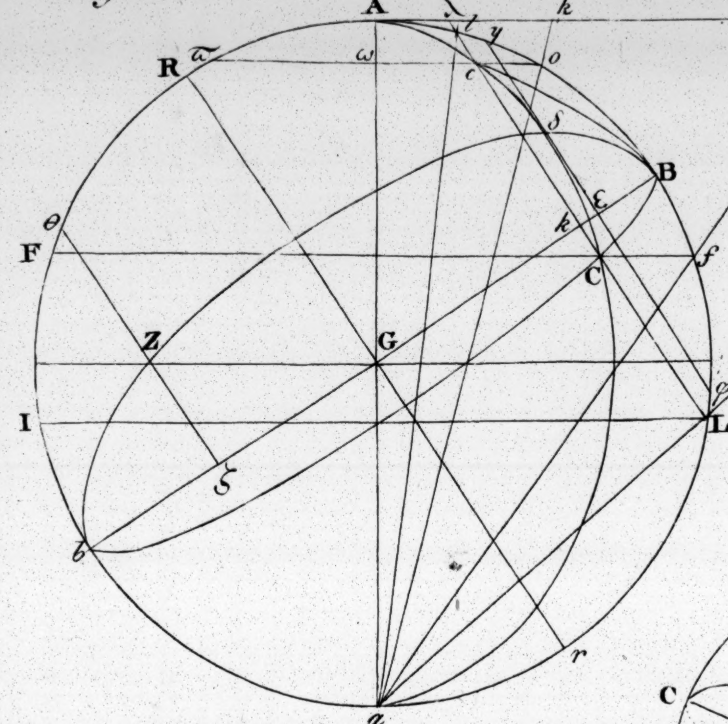

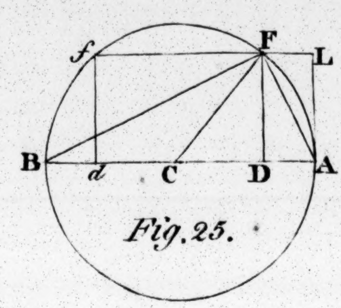

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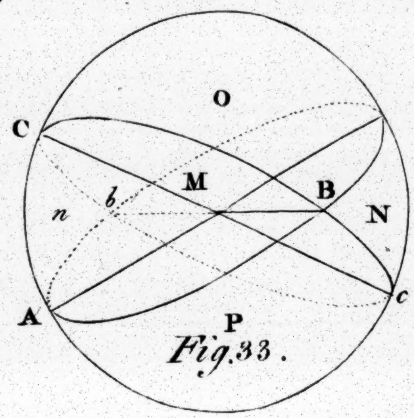


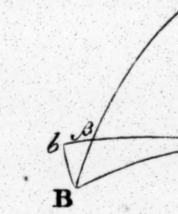
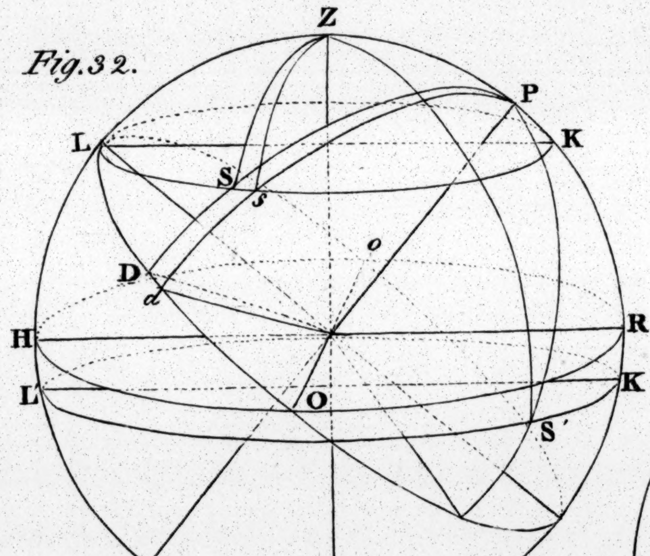
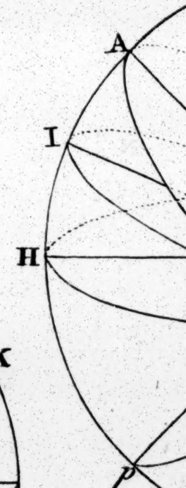
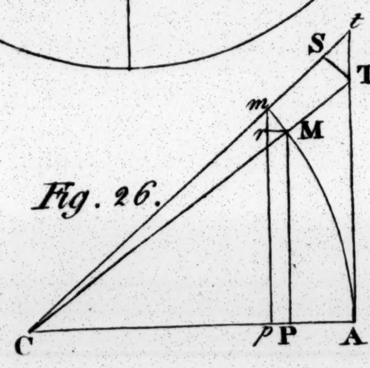


Fig.33.

Fig.32.







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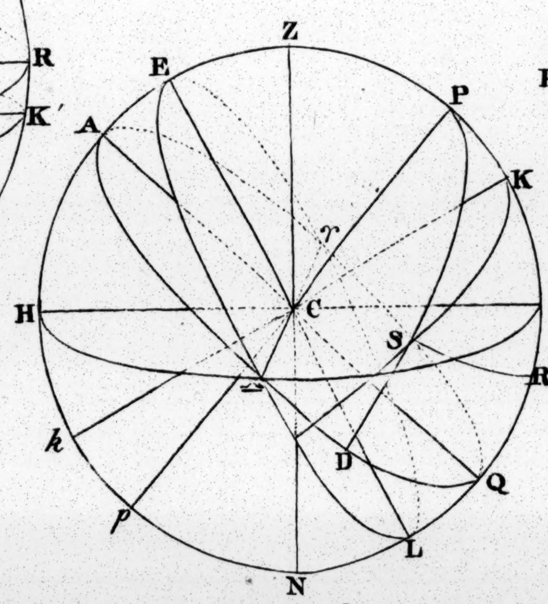
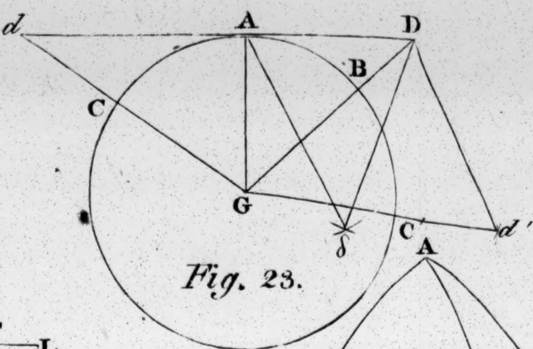


Fig.36.

Pl. II at the End.



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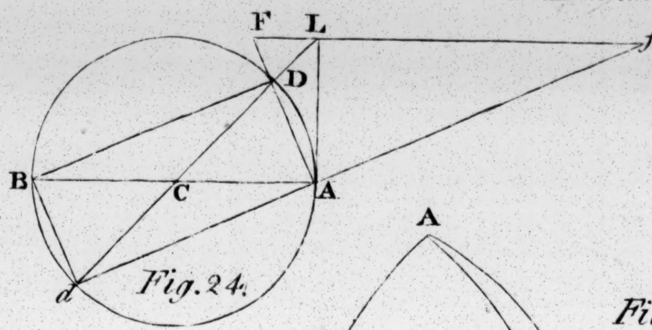

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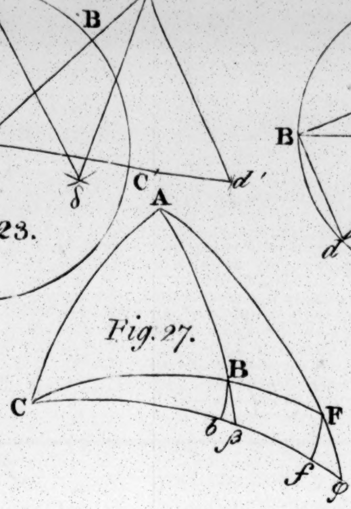

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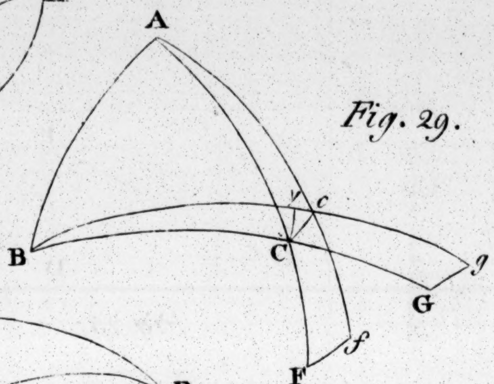

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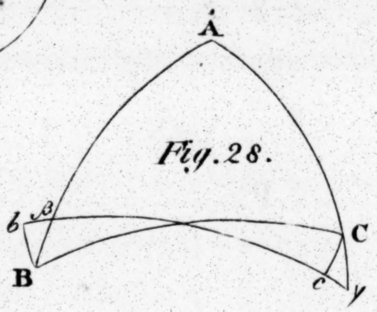

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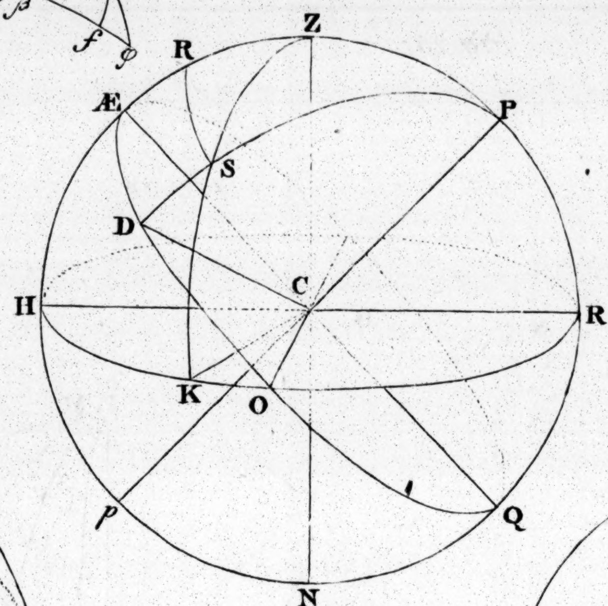

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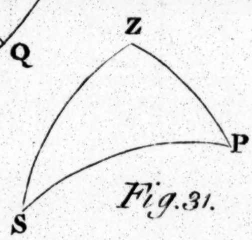

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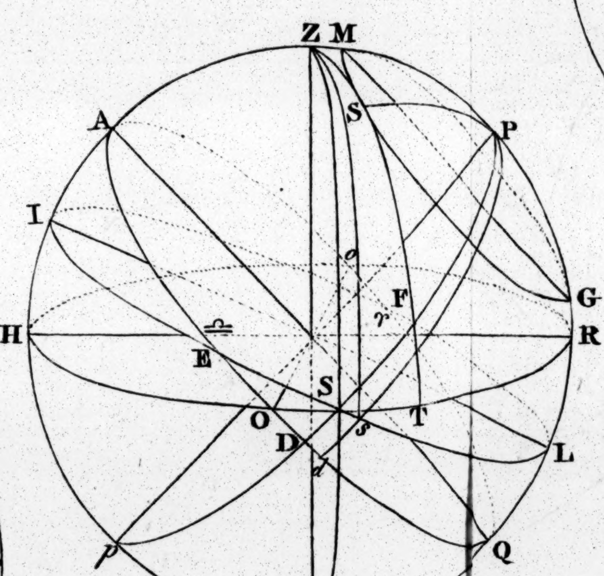

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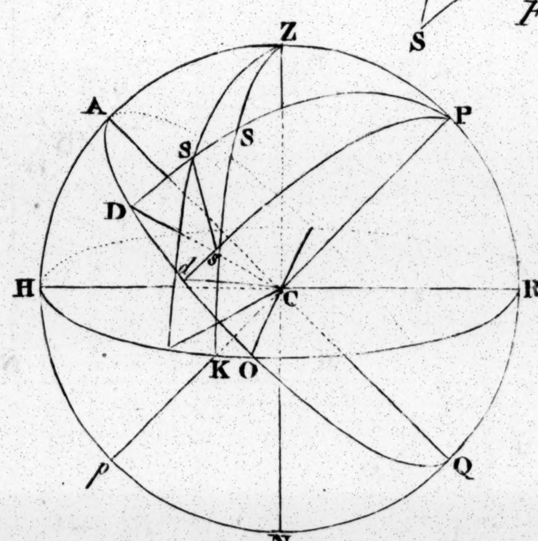

Fig. 37.

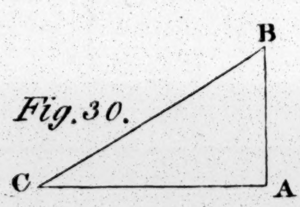

Fig. 30.